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String primer

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Abstract: This is the written version of a set of introductory lectures to string theory. The lectures were given at the Universidad Autónoma de Madrid in the semester 1997/98 and at the *VI Escuela de Otoño de Física Teórica*, held in Santiago de Compostela (10-23 september 1998).

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1. Motivation

Strings, in a broad sense, is a topic studied by a sizeable fraction of the particle physics community since the mid-eighties. In this interval it has gotten the reputation, among some, of belonging to the limbo of unfalsifiable theories, sharing this place with Inflation, Quantum Gravity *et cetera*.

To date it is difficult to argue that phenomenological predictions are around the corner and it is fair to say that there does not yet appear any physically appealing guiding principle (something similar to the equivalence principle and general covariance in General Relativity) in the new developments.

And yet, the new developments are fascinating. There is a renewed (and deeper) sense in which it can be claimed that all five string theories are manifestations of some unique *M-theory*, described at long wavelengths by 11-dimensional supergravity. Conformal Field Theory, and String Perturbation Theory are now stored waiting for a corner of parameter space in which they could make useful physical predictions. In a sense, the situation has some similarities with the late seventies, when the non-perturbative structure of the QCD vacuum started to being appreciated. In strings, non-perturbative effects are known to be important (in particular, all sorts of extended objects spanning p spacelike dimensions, *p-branes*), and plenty of astonishing consistency checks can be made, without meeting any clear contradiction (so far). To the already seemingly miraculous correlation between world-sheet and spacetime phenomena, one has to add, no less surprising, interrelations between physics on the world-volume of a D(irichlet)-brane (described by supersymmetric Yang-Mills) and physics on the bulk of spacetime (including gravity).

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Many properties of supersymmetric *field* theories can easily be understood by engineering appropriate brane configurations. Also, the classical string relationship *closed* = *open* × *open* seems to be valid, at least for S-matrix elements, also for field theory, in the sense that *gravity* = *gauge* × *gauge* [1].

The implementation of the Montonen-Olive conjecture by Seiberg and Witten in theories with only N=2 supersymmetry led to the first concrete Ansatz embodying confinement in field theory to date.

Unfortunately, in many aspects the situation is even worse than in QCD. The structure of the vacuum and the symmetries of the theory are still unknown. The status of p-branes with respect to quantum mechanics is still unclear for $p > 1$. That is, it is not known whether membranes and higher branes are fundamental objects to be quantized, or only passive topological defects on which strings (corresponding to $p = 1$) can end. Besides, the amount of physical observables which can be computed has not increased much with respect to the pre-duality period.

Still, it can be said, paraphrasing Warren Siegel [2], that this is *the best time for someone to read a book on the topic and the worst time for someone to write one.* (He presumably meant it to encourage people to work in open topics such as this one). The aim of these lectures (written under duress) is quite modest: to whet the appetite of some students for these matters, and to direct them to the study of the original papers, or at least, to books and reviews written by the authors who made the most important contributions, many of them cited in the bibliography [2]–[25].

2. Maximal supergravity, p-branes and electric/magnetic duality for extended objects

It has been emphasized many times before why supersymmetry is a fascinating possibility. Besides being the biggest possible symmetry of the S-matrix (the Haag-Łopuszanski-Sohnius theorem), it can solve many phenomenological *naturalness* problems on the road to unification, and, at the very least, provide very simple (*i.e.* finite) quantum field theories (the analogue of the harmonic oscillator in quantum mechanics) from which more elaborate examples could hopefully be understood.

Supergravities in all possible spacetime dimensions have been classified by W. Nahm [26]. The highest dimension in which it is possible to build an action with highest spin two[1] is ($N = 1$ supergravity in) $d = 11$, and this was done in a classic paper by Cremmer, Julia and Scherk [27]. Upon (toroidal) dimensional reduction this theory leads to $N = 8$ supergravity in $d = 4$, giving in the process a set of theories in different dimensions with 32 (real) supercharges.

[1] It is not known how to write down consistent actions with a finite number of fields containing spin 5/2 and higher.

Giving the fact that this is, in a sense, the most symmetric of all possible theories we can write down, let us examine the hypothesis that it is also the most fundamental, in a sense still to be clarified.

2.1 $N = 1$ supergravity in 11 dimensions

The action can be written as

$$S = \int d^{11}x \left\{ -\frac{e}{4\kappa^2} R(\omega) - i\frac{e}{2}\bar{\psi}_M \Gamma^{MNP} D_N(\frac{\omega + \hat{\omega}}{2})\psi_P - \right. \tag{2.1}$$
$$- \frac{e}{48} F_{MNPQ} F^{MNPQ} + \frac{2\kappa}{(144)^2} \epsilon^{A_1 \ldots A_{11}} F_{A_1 \ldots A_4} F_{A_5 \ldots A_8} A_{A_9 \ldots A_{11}} +$$
$$\left. + \frac{\kappa e}{192} \left(\bar{\psi}_{A_1} \Gamma^{A_1 \ldots A_6} \psi_{A_2} + 12 \bar{\psi}^{A_3} \Gamma^{A_4 A_5} \psi^{A_6}\right) \left(F_{A_3 \ldots A_6} + \hat{F}_{A_3 \ldots A_6}\right) . \right\}$$

Here e is the determinant of the *Elfbein* representing the graviton (with zero mass dimension); ψ_M represents the gravitino (of mass dimension 5), taken as a C_- Majorana[2] Rarita-Schwinger vector-spinor (a Majorana spinor in $D = 11$ has 32 real components); and A_{MNP} is a (mass dimension $\frac{9}{2}$) three-form field, a kind of three-index Maxwell field.

The Lorentz connection is given in terms of the Ricci rotation coefficients and the contorsion tensor as

$$\omega_{Mab} = \omega^{Ricci}_{Mab} + K_{Mab} \,, \tag{2.2}$$

and the contorsion tensor itself is given by

$$K_{Mab} = \frac{i\kappa^2}{4} \left(-\bar{\psi}_\alpha \gamma^{\alpha\beta}_{Mab} \psi_\beta + 2 \left(\bar{\psi}_M \gamma_b \psi_a - \bar{\psi}_M \gamma_a \psi_b + \bar{\psi}_b \gamma_\mu \psi_a \right) \right) . \tag{2.3}$$

The *supercovariant* connection and field strength are given by

$$\hat{\omega}_{Mab} \equiv \omega_{Mab} + \frac{i\kappa^2}{4} \bar{\psi}_\alpha \gamma^{\alpha\beta}_{Mab} \psi_\beta \,, \tag{2.4}$$

and

$$\hat{F}_{MNPQ} = F_{MNPQ} - 3\kappa \bar{\psi}_{[M} \gamma_{NP} \psi_{Q]} \,. \tag{2.5}$$

On shell, the graviton corresponds to the (2,0,0,0) representation of the little group SO(9), with 44 real states; the three-form to the (1,1,1,0) of SO(9), with 84 real polarization states; and, finally, the gravitino lives in the (1,0,0,1), yielding 128 polarizations which matches the bosonic degrees of freedom.

The Chern-Simons-like coupling in the preceding action suggests a 12-dimensional origin but, in spite of many attempts, there is no clear understanding of how this could come about.

[2] We define, following [28], $C_\pm$ Majorana spinors as those obeying $\psi^T C_\pm = \alpha \psi^+ \gamma_0$, with $\gamma^T_\mu = \pm C_\pm \gamma_\mu C^{-1}_\pm$, and α is a phase.

There are a couple of further remarkable properties of this theory (stressed, in particular by Deser [29]). First of all, there is no globally supersymmetric matter (with highest spin less than 2), which means that there are no sources. Furthermore, it is the only theory which forbids a cosmological constant because of a symmetry (i.e. it is not possible to extend the theory to an Anti-de Sitter background, although this is an active field of research). Let us now concentrate on the three-form $A_{(3)} \equiv \frac{1}{3!}A_{MNP}dx^M \wedge dx^N \wedge dx^P$. From this point of view, the Maxwell field is a one-form $A_{(1)} \equiv A_M dx^M$, which couples minimally to a point particle through

$$e \int_\gamma A \,, \tag{2.6}$$

where the integral is computed over the trajectory $\gamma : x^\mu = x^\mu(s)$ of the particle. A Particle is a zero dimensional object, so that its world-line has one dimension more, that is, it is a one-dimensional world-line. It is very appealing to keep the essence of this coupling in the general case, so that a general (p+1)-form would still couple in exactly the same way as before, except that now γ must be a (p+1)-dimensional region of spacetime. If we want to interpret this region as the world-volume of some object, it would have to be a p-dimensional extended object, a *p-brane.*

In this way we see that just by taking seriously the geometrical principles of minimal coupling we are led to postulate the existence of *two-branes* (*membranes*) naturally associated to the three-form of supergravity.

On the other hand, as has been stressed repeteadly by Townsend [30], the maximally extended (in the sense that it already has 528 ($= \frac{32\times 33}{2}$) algebraically independent charges, the maximal amount possible) supersymmetry algebra in $d = 11$ is

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)P_M + (C\Gamma_{MN})_{\alpha\beta}Z^{MN}_{(2)} + (C\Gamma_{M_1\ldots M_5})_{\alpha\beta}Z^{M_1\ldots M_5}_{(5)} \,. \tag{2.7}$$

Clearly the first term on the r.h.s. would be associated to the graviton, the second one to the membrane, and the last one to the fivebrane.

This fact (given our present inability to quantize branes in a consistent way) in turn suggests that 11-dimensional supergravity can only be, at best, the long wavelength limit of a more fundamental theory, dubbed *M-Theory.* We shall return to this point later on.

2.2 The Dirac monopole

Many of the properties of charged extended objects are already visible in the simplest of them all: Dirac's magnetic monopole in ordinary four dimensional Maxwell theory (cf. [31, 32]). Although Dirac's magnetic monopole is pointlike, we shall see that one needs to introduce an extended object in order to have a gauge-invariant description of it.

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We *assume* that there is a pointlike magnetic monopole, with magnetic field given by

$$\vec{B}_m \equiv \frac{g}{4\pi r^2}\hat{r} \tag{2.8}$$

($\hat{r} \equiv \frac{\vec{r}}{r}$). In quantum mechanics, minimal coupling demands the existence of a vector potential $\vec{A}$ such that $\vec{B}_m = \vec{\nabla} \times \vec{A}_m$. Unfortunately, this is only possible when $\vec{\nabla} \cdot \vec{B}_m = 0$, which is not the case, but rather $\vec{\nabla} \cdot \vec{B}_m = g\delta^3(x)$. Dirac's way out was to introduce a *string*, (along the negative z-axis, although its position is a gauge-dependent concept) with magnetic field $\vec{B}_s = g\theta(-z)\delta(x)\delta(y)\hat{z}$, such that the total magnetic field $\vec{B}_m + \vec{B}_s$ is divergence-free.

It is quite easy to compute the vector potential of the monopole, $\vec{A}_m$. The flux through the piece of the unit sphere with polar angle, parametrized by θ' say, smaller than θ, which will be called $S_{(+)}$ is given by (using Stokes' theorem and the spherically symmetric *Ansatz*, $\vec{A}_m = A(r,\theta)\,\partial_\phi$),

$$\Phi(S_{(+)}) = \int_{S_{(+)}} \vec{B_m}.d\vec{S} = \int_{C\equiv\partial S_{(+)}} \vec{A}_m.d\vec{l} = 2\pi A r^2 \sin^2\theta\,. \tag{2.9}$$

On the other hand, knowing that the total flux through the sphere is $4\pi g$, we could write $\Phi(S_{(+)})$ as the solid angle subtended by $S_{(+)}$,

$$\Phi(S_{(+)}) = \frac{g}{4\pi}\Omega(S_{(+)}) = \frac{g}{4\pi}\int_0^\theta d\theta' \sin\theta' d\phi' = \frac{g}{2}(1-\cos\theta)\,. \tag{2.10}$$

This yields

$$\vec{A}_m^{(+)} = \frac{g}{4\pi r^2 \sin^2\theta}(1-\cos\theta)\partial_\phi\,. \tag{2.11}$$

There is a certain ambiguity because of $\partial S_{(+)} = \partial S_{(-)}$, where $S_{(-)}$ is the complementary piece of the unit sphere defined by $\theta'' > \theta$. Using Stokes theorem on the lower piece, the flux is given by

$$\begin{aligned}\Phi_{(-)}(C) &= \left(1 - \frac{\Omega_{(+)}}{4\pi}\right) g \\ &= \frac{g}{2}(1+\cos\theta) \\ &= -\int_{\partial S_{(-)}} \vec{A_m} d\vec{l},\end{aligned} \tag{2.12}$$

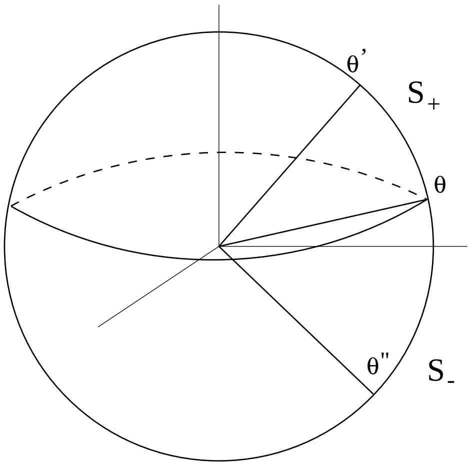

Figure 1: Sphere surrounding a Dirac monopole

yielding

$$\vec{A}_m^{(-)} = -\frac{g}{4\pi r^2 \sin^2\theta}(1+\cos\theta)\,\partial_\phi\,. \tag{2.13}$$

In both cases $\vec{\nabla}\times\vec{A}_m^{(+)} = \vec{\nabla}\times\vec{A}_m^{(-)} = \frac{g}{4\pi r^2}\hat{r}$. The corresponding covariant vectors, expressed as one-forms, are

$$A_m^{(\pm)} = \frac{g}{4\pi}\frac{1}{2r}\frac{xdy - ydx}{z\pm r} = \frac{g}{4\pi}(\pm 1 - \cos\theta)d\phi\,. \tag{2.14}$$

In this language it is obvious that the two possible determinations of the gauge potential of the monopole differ by a *gauge* transformation

$$A^{(+)} - A^{(-)} = d\Lambda \equiv -\frac{g}{2\pi}d\phi\,. \tag{2.15}$$

Had we included the string in the computation, the $A^{(+)}$ would remain unaffected, and $\vec{A}_{m-s}^{(-)} = \vec{A}_{m-s}^{(+)} = \vec{A}_m^{(+)}$,[3] such that the one-form potential assotiated to the string in the said configuration is the closed, but not exact one-form $A_s \equiv \frac{g}{2\pi}d\phi$.

Demanding that the gauge transformation connecting the two potentials is single valued acting on fields minimally charged, that is $e^{ie\Lambda(\phi=0)} = e^{ie\Lambda(\phi=2\pi)}$ imposes Dirac's quantization condition

$$\frac{eg}{2\pi}\in\mathbb{Z}\,. \tag{2.16}$$

2.3 Extended poles

The only purely geometrical action for a $(p-1)$-brane with a classical trajectory, parametrized by

$$X^\mu \;=\; X^\mu\left(\xi^0,\ldots,\xi^{p-1}\right), \tag{2.17}$$

is the p-dimensional world-volume induced on the trajectory by the external d dimensional metric

$$\mathcal{S} = -T_p\int_{W_p} d\,(Vol)\,, \tag{2.18}$$

where the Riemannian volume element is given in terms of the determinant of the world-volume metric $h \equiv \det(h_{ij})$, by

$$d\,(Vol) = d\xi^{p-1}\wedge\ldots\wedge d\xi^0\,\sqrt{|h|}\,. \tag{2.19}$$

The metric on the world-volume is the one induced from the spacetime metric by the imbedding itself, namely

$$h_{ij} = \partial_i X^\mu\,\partial_j X^\nu\,g_{\mu\nu}(X)\,. \tag{2.20}$$

Classically, this is equivalent to the Polyakov-type action

$$\mathcal{S} \;=\; T_p\int_{W_p} d^p\xi\,\sqrt{|h|}\left[-\frac{1}{2}h^{ij}\partial_i X^\mu\partial_j X^\nu g_{\mu\nu}(X) \;+\; \frac{1}{2}(p-2)\right]. \tag{2.21}$$

[3] It should be clear that this whole argument fails at the origin

In this action the two-dimensional metric is now a dynamical field as well as the imbeddings. On shell, the equations of motion for the two-dimensional metric force it to be equal to the induced metric (2.20), but off-shell this is not the case.

Given an external p-form field A_p, there is a natural ("Wess-Zumino") coupling to the $(p-1)$-brane

$$S_{int} = e_p \int_{W_p} \tilde{A}_p \,, \tag{2.22}$$

where the induced form on the world-volume is given by

$$\tilde{A}_p = \frac{1}{p!} A_{\mu_1 \ldots \mu_p}(X)\, \partial_{i_1} X^{\mu_1} \ldots \partial_{i_p} X^{\mu_p}\, d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_p} \,. \tag{2.23}$$

In any *Abelian* theory of p-forms with gauge invariance

$$A_p \to A_p + d\Lambda_{p-1} \,, \tag{2.24}$$

the standard definition of the field strength $F_{p+1} \equiv dA_p$, implies the *Bianchi identity*, to wit

$$dF_{p+1} = 0 \,. \tag{2.25}$$

The minimal "Maxwell" action for Abelian p-forms is

$$\mathcal{S} = \int_{V_d} d\,(Vol)\,(*F)_{d-p-1} \wedge F_{p+1} \,. \tag{2.26}$$

The equations of motion of the p-form itself can be written as

$$\frac{\delta S}{\delta A_p} = d(*F)_{d-p-1} = \left(*J^{(e)}\right)_{d-p} \,, \tag{2.27}$$

where the source $J^{(e)}$ is a (p)-form with support in W_p, the world-volume spanned by the (p-1)-brane,

$$J^{(e)}_p = e_p \left\{ \int_{W_p} \frac{1}{p!} \partial_{i_1} X^{\mu_1} \ldots \partial_{i_p} X^{\mu_p} d\xi^{i_1} \ldots d\xi^{i_p} \right\} dX_{\mu_1} \wedge \ldots \wedge dX_{\mu_p} \,. \tag{2.28}$$

Electric charges are naturally defined as a boundary contribution in the subspace orthogonal to the world-volume, $M^{\perp}_{d-p}$ (in which all the history of the brane is just a point),

$$e_p \equiv \int_{S_{d-p-1} = \partial M^{\perp}_{d-p}} (*F)_{d-p-1} = \int_{M^{\perp}_{d-p}} \left(*J^{(e)}\right)_{d-p} \,. \tag{2.29}$$

This means that $(*J^{(e)})_{d-p}$ is a Dirac current with support in $M^{\perp}_{d-p}$, with charge e_p.

In ordinary Maxwell theory,

$$A_1 = \frac{e_1}{4\pi r} dt \,, \tag{2.30}$$

so that

$$F_2 \equiv dA_1 = \frac{e_1}{4\pi r^3} \sum_i x^i dt \wedge dx^i \,. \tag{2.31}$$

The Hodge dual is given by

$$(*F)_2 = \frac{e_1}{4\pi r^3} \sum_{i,j,k} x^i \epsilon_{ijk} dx^j \wedge dx^k \,, \tag{2.32}$$

and indeed

$$d(*F)_2 = e_1 \delta^{(3)}(x) d(\mathrm{vol}) \,, \tag{2.33}$$

so that

$$J_1^{(e)} = e_1 \delta^{(3)}(x) dt \,. \tag{2.34}$$

The string is geometrically the place on which there fails to exist a potential for $*F$, (when there is an electric source) because if we write

$$(*F)_{d-p-1} \equiv d\tilde{A}_{d-p-2} + (*\tilde{S})_{d-p-1} \,, \tag{2.35}$$

consistency with the equations of motion demands that

$$d(*\tilde{S})_{d-p-1} = \left(*J^{(e)}\right)_{d-p} \,. \tag{2.36}$$

Given a $(p-1)$-brane, then, coupling to an A_p, the *dual brane*, coupling to $\tilde{A}_{d-p-2}$ will be a $(\tilde{p} \equiv d-p-3)$-brane. For example, in $d=4$ the dual of a 0-brane is again a 0-brane. In $d=11$, however, the dual of a 2-brane is a 5-brane.

We would like to generalize Dirac's construction to this case. This would mean introducing a magnetic source such that

$$dF_{p+1} = J^{(m)}_{p+2} \,, \tag{2.37}$$

(and we do not have now an electric source, so that $d*F=0$) which is incompatible with the Bianchi identity, unless we change the definition of F_{p+1}. In this way we are led to (re)define

$$F_{p+1} \equiv dA_p + S_{p+1} \,, \tag{2.38}$$

where the Dirac (hyper)string is an object, with a $(p+1)$-dimensional world-volume, such that

$$dS_{p+1} = J^{(m)}_{p+2} \,. \tag{2.39}$$

Under these conditions the magnetic charge is defined as

$$g_{d-p-2} = \int_{\partial M_{p+2}} F_{p+1} = \int_{M_{p+2}} J^{(m)}_{p+2} \,. \tag{2.40}$$

This means that $J^{(m)}_{p+2}$ is a Dirac current with support in M_{p+2} and charge g_{d-p-2}.

Writing the (free) action without any coupling constant, the form field has mass dimension $[A_p] = \frac{d}{2} - 1$, which implies that $[e_p] = p + 1 - \frac{d}{2}$, and $[g_{d-p-2}] = \frac{d}{2} - p - 1$.

Demanding now that the (hyper)string could not be detected in a Böhm-Aharanov experiment using a (d-p-3)-brane imposes that the phase factor it picks up when it moves around the string is trivial [33],

$$\exp ie_p \int F_{p+1} = \exp ie_p g_{d-p-2}\,, \tag{2.41}$$

i.e.

$$e_p g_{d-p-2} \in 2\pi\mathbb{Z}\,. \tag{2.42}$$

2.4 Bogomol'nyi-Prasad-Sommerfeld states

In extended SUSY it is possible for some central charges to enter the commutation relations between the supercharges, as predicted by the Haag-Łopuszanski-Sohnius theorem. The supersymmetry relations in that case put a restriction on the lowest value for the energy in terms of the eigenvalues of this central charge. When this bound is saturated, the states are called *BPS states*, and they are stable by supersymmetry. Also, those states form supersymmetry multiplets of dimension lower than non-BPS states, the so-called *short multiplets*. This means that most of their physical properties, like masses, charges etc., are protected from quantum corrections and can be computed at lowest order in perturbation theory. Physically, this bound in the most important cases takes the form of $M \geq kQ$, where M is the mass of the state, k is a parameter of order unity, and Q is a geometric mean of the charges of the said state.

Let us illustrate all this with a very simple quantum mechanical example due to Polchinski [9]. We are given two charges, such that the commutation relations read

$$\left\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\right\} = H + Z\,, \tag{2.43}$$

$$\left\{\mathcal{Q}_2, \mathcal{Q}_2^\dagger\right\} = H - Z\,, \tag{2.44}$$

where Z is a central charge, i.e. it commutes with all the elements of the SUSY algebra.

There is a 4-state representation in a given (h, z) sector of the four-dimensional Fock space

$$\begin{aligned} &\mathcal{Q}_1 \mid 0\,0\rangle = 0\,, && \mathcal{Q}_2 \mid 0\,0\rangle = 0\,, \\ &\mathcal{Q}_1^\dagger \mid 0\,0\rangle = \lambda_1 \mid 1\,0\rangle\,, && \mathcal{Q}_2^\dagger \mid 0\,0\rangle = \lambda_2 \mid 0\,1\rangle\,. \end{aligned} \tag{2.45}$$

and, of course,

$$\mathcal{Q}_2^\dagger \mid 1\,0\rangle = \lambda_3 \mid 1\,1\rangle \tag{2.46}$$

By hypothesis we see that

$$h + z = \langle 0\,0 \mid \mathcal{Q}_1 \mathcal{Q}_1^\dagger \,+\, \mathcal{Q}_1^\dagger \mathcal{Q}_1 \mid 0\,0 \rangle \;=\; \langle 1\,0 \mid 1\,0 \rangle\, |\lambda_1|^2 \,, \tag{2.47}$$
$$h - z = \langle 0\,0 \mid \mathcal{Q}_2 \mathcal{Q}_2^\dagger \,+\, \mathcal{Q}_2^\dagger \mathcal{Q}_2 \mid 0\,0 \rangle \;=\; \langle 0\,1 \mid 0\,1 \rangle\, |\lambda_2|^2 \,, \tag{2.48}$$
$$h - z = \langle 1\,0 \mid \mathcal{Q}_2 \mathcal{Q}_2^\dagger \,+\, \mathcal{Q}_2^\dagger \mathcal{Q}_2 \mid 1\,0 \rangle \;=\; \langle 1\,1 \mid 1\,1 \rangle\, |\frac{\lambda_3}{\lambda_1}|^2 \,, \tag{2.49}$$

and it should be obvious that $h \geq |z|$. Note that when $h = |z|$, $\lambda_2 = \lambda_3 = 0$ and we have a two-state representation, generated by $\mid 0\,0\rangle$ and $\mid 1\,0\rangle$, because $Q_2^\dagger \mid 0\,0\rangle \;= 0 = Q_1^\dagger Q_2^\dagger \mid 0\,0\rangle \;= -Q_2^\dagger \lambda_1 \mid 1\,0\rangle$.

The *number* of BPS states is a sort of topological invariant, which does not change under smooth variations of the parameters of the theory (like coupling constants). This fact is at the root of the recent successes in counting the states corresponding to configurations which are in a sense equivalent to extremal black holes (see section 6.3).

2.5 Brane surgery

P. K. Townsend [30] has shown how to get information on intersections of branes (i.e. which type of brane can end on a given brane) by a careful examination of the Chern-Simons term in the action.

In $d = 11$ the 2-brane carries an electric charge

$$Q_2 = \int_{S^7} (*F)_7 \,, \tag{2.50}$$

where S^7 is a sphere surrounding the brane in the 8-dimensional transverse space (in which the brane is just a point).

The analogous expression for the 5-brane is

$$Q_5 = \int_{S^4} F_4 \,. \tag{2.51}$$

The Bianchi identity for F is $dF = 0$, meaning that charged 5-branes must be closed (otherwise one could slide off the S^4 encircling the brane, and contract it to a point, the Bianchi identity guaranteeing that the integral is an homotopy invariant).

This argument does not apply to the 2-brane, however, owing to the presence of the Chern-Simons term in the 11-dimensional supergravity action, which modifies the dual Bianchi identity to

$$d * F = -F \wedge F \,. \tag{2.52}$$

This fact implies that the homotopy invariant charge is

$$\tilde{Q}_2 \equiv \int_{S^7} *F + F \wedge A \,. \tag{2.53}$$

If the 2-brane had a boundary, the last term could safely be ignored as long as the distance L from the boundary to the S^7 is much bigger than the radius R of the sphere itself. If now we slide the S^7, keeping L/R large as $L \to 0$, as $L = 0$ the sphere collapses to the endpoint (which must be assignated to a nonvanishing value of the Chern-Simons, if a contradiction is to be avoided). At this stage the sphere can be deformed to the product $S^4 \times S^3$, in such a way that the contribution to the charge is

$$\tilde{Q}_2 = \int_{S^4} F \int_{S^3} A \,. \tag{2.54}$$

The first integral is the charge Q_5 associated to a 5-brane. Choosing also $F_{\|} = 0$ (so that $A = dV_2$ in the second integral), we would have

$$\tilde{Q}_2 = Q_5 \int_{S^3} dV_2 \,, \tag{2.55}$$

namely, the (magnetic) charge of the string boundary of the 2-brane in the 5-brane.

We have learned from the preceding analysis that in $d = 11$ a 2-brane can end in a 5-brane, with a boundary being a 1-brane. A great wealth of information can be gathered by employing similar reasonings to 10-dimensional physics.

2.6 Dyons, theta angle and the Witten effect

There are allowed configurations with both electric and magnetic charge simultaneously, called *dyons*, whose charges will be denoted by (e, g). Given two of them, the only possible generalization of Dirac's quantization condition compatible with electromagnetic duality, called the Dirac-Schwinger-Zwanziger quantization condition, is [31]

$$e_1 g_2 - g_1 e_2 = 2\pi \mathbb{Z} \,. \tag{2.56}$$

E. Witten [34] pointed out that in the presence of a theta term in the Yang-Mills action, the electric charges in the monopole sector are shifted. There is a very simple argument by Coleman, which goes as follows: the theta term in the Lagrangian can be written as

$$-\frac{\theta e^2}{32\pi^2} F^a_{\mu\nu} * F^{a\mu\nu} \,, \tag{2.57}$$

which for an Abelian configuration reduces to

$$\frac{\theta e^2}{8\pi^2} \vec{E} \cdot \vec{B} \,. \tag{2.58}$$

In the presence of an magnetic monopole one can write

$$\begin{aligned} \vec{E} &= \vec{\nabla} A_0 \,, \\ \vec{B} &= \vec{\nabla} \times \vec{A} + \frac{g}{4\pi} \frac{\vec{r}}{r^2} \,, \end{aligned} \tag{2.59}$$

which when used in the action, yields

$$\delta S = \frac{\theta e^2}{8\pi^2}\int d^3x \vec{\nabla} A_0 \left(\vec{\nabla}\times\vec{A} + \frac{g}{4\pi}\frac{\vec{r}}{r^2}\right)$$
$$= -\frac{\theta e^2 g}{8\pi^2}\int d^3x A_0 \delta^3(x)\,. \tag{2.60}$$

This last term is nothing but the coupling of the scalar potential A_0 to an electric charge $-\frac{\theta e^2 g}{8\pi^2}$ at $x = 0$. This means that a minimal charge monopole[4] with $eg = 4\pi$ has an additional electric charge $-\frac{e\theta}{2\pi}$.

All this means that the explicit general solution to the quantization condition in the presence of a theta term is

$$Q_m = \frac{4\pi n_m}{e}\,,$$
$$Q_e = n_e e - \frac{e n_m \theta}{2\pi}\,. \tag{2.61}$$

Montonen and Olive [36] proposed that in a non-Abelian gauge theory (specifically, in an SO(3) Yang-Mills-Higgs theory) there should exist (at least in the BPS limit) an *exact* duality between electric and magnetic degrees of freedom.

It was soon realized by Osborn [37] that, in order for this idea to have any chance to be correct, supersymmetry was necessary, and the simplest candidate model was $N = 4$ supersymmetric Yang-Mills in 4 dimensions, with Lagrangian given by

$$L = -\frac{1}{4}Tr\left(F_{\mu\nu}F^{\mu\nu}\right) + i\lambda_i\sigma^\mu D_\mu\bar{\lambda}^i + \frac{1}{2}D_\mu\Phi_{ij}D^\mu\Phi^{ij} +$$
$$+ i\lambda_i[\lambda_j,\Phi^{ij}] + i\bar{\lambda}^i[\bar{\lambda}^j,\Phi_{ij}] + \frac{1}{4}[\Phi_{ij},\Phi_{kl}][\Phi^{ij},\Phi^{kl}]\,. \tag{2.62}$$

where the gauginos are represented by four Weyl spinors λ_i, transforming in the **4** of SO(6), and the six scalar fiels Φ_{ij} obey $(\Phi_{ij})^\dagger \equiv \Phi^{ij} = \frac{1}{2}\epsilon^{ijkl}\Phi_{kl}$.

By defining the parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\,, \tag{2.63}$$

where in our case $\theta = 0$ and the coupling constant has been absorbed in the definition of the gauge field, electric-magnetic duality (dubbed *S-duality* in this context) would be an $SL(2,\mathbb{Z})$ symmetry

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{2.64}$$

where $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. Please note that this is a *strong-weak* type of duality, because the particular element $\tau \to -\frac{1}{\tau}$ transforms (when $\theta = 0$, for simplicity) $\alpha \equiv \frac{e^2}{4\pi}$ into $\frac{1}{\alpha}$.

[4]Note that due to the possibility of coupling the theory to fields having half-integer charges, $e/2$, the Dirac quantization condition reads $eg = 4\pi\mathbb{Z}$ [35].

S-duality is believed to be an exact symmetry of the full quantum field theory. In spite of the fact that this theory is conformal invariant ($\beta = 0$) and is believed to be finite in perturbation theory, the hard evidence in favour of this conjecture is still mainly kinematical (cf. Vafa and Witten in [38]). (Dynamical interactions between monopoles are notoriously difficult to study beyond the simplest approximation using geodesics in moduli space [39]).

This S-Duality in Quantum Field Theory is closely related to a corresponding symmetry in String Theory, also believed to be exact for toroidally compactified heterotic strings as well as for ten dimensional Type IIB Strings.

2.7 Kappa-symmetry, conformal invariance and strings

It is not yet known what is to be the fundamental symmetry of fundamental physics. For all we know, however, both kappa symmetry and conformal invariance are basic for the consistency of any model, specifically for spacetime supersymmetry and for the absence of anomalies.

2.7.1 Kappa-symmetry

When considering *branes*, the fact that those p-branes are embedded in an *external* spacetime M_d

$$\xi^i \to X^\mu(\xi^i)\,, \tag{2.65}$$

where $i = 0, \ldots, p$ and $\mu = 0, \ldots, (d-1)$, is the root of an interesting interplay between *world-volume* properties (i.e. properties of the theory defined on the brane, where the X^μ are considered as fields, with consistent quantum properties when $p = 1$) and spacetime properties, that is, properties of fields living on the target-space, whose coordinates are the X^μ themselves.

One of the subtler aspects of this correspondence is the case of the fermions. If the target-space (*spacetime*) theory is supersymmetric, the most natural thing seems to use fields which are spacetime fermions to begin with. This is called the *Green-Schwarz* kind of actions.

It so happens that it is also possible to start with world-sheet fermions, which are spacetime vectors, and then reconstruct spacetime fermions through bosonization techniques (the Frenkel-Kač-Segal construction). This is called the *Neveu-Schwarz-Ramond* action.

The first type of actions are only imperfectly understood and, in particular, it is not known how to quantize them in a way which does not spoil manifest covariance. This is the reason why the NSR formalism is still the only systematic way to perform string perturbation theory.

There is however one fascinating aspect of Green-Schwarz actions worth mentioning: they apparently need, by consistency the presence of a particular fermionic symmetry, called κ-*symmetry* which allows to halve the number of propagating fermionic degrees of freedom.

It was apparently first realized by Achúcarro et al. [40] that in order to get a consistent theory one needs world-sheet supersymmetry realized linearly, that is, with matching fermionic and bosonic degrees of freedom (d.o.f.). The condition for equal number of bosonic and fermionic d.o.f. after halving the (real) fermionic components of the minimal spinor,[5] is [40], $N_S^{SUSY} \times \frac{1}{2}\,(\kappa) \times \tilde{d}_F = d - p - 1$.

For example:

- $p = 0$ (Particles): applying the above formulas one finds that $N_S = 4$ for $d = 2$ and $N_S = 3$ when $d = 4$.

- $p = 1$ (Strings): for $d = 10$ one finds that $N_S = 2$, corresponding to the type IIA and type IIB theories.

- $p = 2$ (Membranes): for $d = 11$ one finds that $N_S = 1$, which is kosher.

It seems to be generally true that exactly,

$$N_{SUSY}(\text{World-volume}) \;=\; \frac{1}{2} N_S(\text{Spacetime})\,. \tag{2.66}$$

To illustrate this idea in the simplest context, consider the Lorentz-invariant action for the superparticle given by

$$S = \frac{1}{2}\int d\tau \frac{1}{e}\left(\dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A\right)\eta_{\mu\nu}\left(\dot{x}^\nu - i\bar{\theta}^A\Gamma^\nu\dot{\theta}^A\right). \tag{2.67}$$

This action is supersymmetric in any dimension without assuming any special reality properties for the target-space spinors $\theta^A(\tau)$: the explicit rules are

$$\delta\theta^A = \epsilon^A\,, \tag{2.68}$$
$$\delta x^\mu = i\bar{\epsilon}^A\Gamma^\mu\theta^A\,. \tag{2.69}$$

Please note that the presence of the *Einbein* e is necessary, because only then the action is reparametrization invariant; i.e. when

$$\tau \to \tau'\,, \tag{2.70}$$
$$e \to e' \equiv \frac{\partial\tau}{\partial\tau'}\,e\,. \tag{2.71}$$

But precisely the equation of motion for e, $\frac{\delta S}{\delta e} = 0$, implies that on-shell the canonical momentum associated to x^μ, $\pi_\mu = \frac{1}{e}(\dot{x}_\mu - i\bar{\theta}^A\Gamma_\mu\dot{\theta}^A)$, is a null vector, $\pi_\mu\pi^\mu = 0$, so

[5] That is $\tilde{d}_F = 2^{[d/2]-1}$ for a Majorana or Weyl spinor, except in $d = 2 + 8\mathbb{Z}$ because one can impose the Majorana and Weyl condition at the same time, so that $\tilde{d}_F = 2^{[d/2]-2}$. For a general discussion of spinors in arbitrary spacetime dimension and signature, the reader is kindly referred to [28].

that as a consequence, the Dirac equation only couples half of the components of the spinors θ^A. This remarkable fact can be traced to the existence of the (κ- or Siegel-)symmetry

$$\delta\theta^A = -ip_\mu\Gamma^\mu\kappa^A\,, \tag{2.72}$$

$$\delta x^\mu = i\bar{\theta}^A\Gamma^\mu\delta\theta^A\,, \tag{2.73}$$

$$\delta e = 4\,e\,\dot{\bar{\theta}}^A\kappa^A\,, \tag{2.74}$$

where κ^A is a (target-space) spinorial parameter. The algebra of κ-transformations closes on shell only, where

$$[\delta(\kappa_1),\delta(\kappa_2)] = \delta(\kappa_{12})\,, \tag{2.75}$$

with $\kappa_{12} \equiv -4\kappa_2\dot{\bar{\theta}}^B\kappa_1^B + 4\kappa_1\dot{\bar{\theta}}^B\kappa_2^B$.

The example of the superparticle is a bit misleading, however, because one always has kappa-symmetry, and this does not impose any restrictions on the spacetime dimensions.

Historically, this kind of symmetry was first discovered in the Green-Schwarz [41] action for the string, by trial and error. Henneaux and Mezincescu [42] interpreted the extra non-minimal term (to be introduced in a moment) as a Wess-Zumino contribution, and Hughes and Polchinski [43] emphasized that the minimal action is of the Volkov-Akulov type, representing supersymmetry nonlinearly in the Nambu-Goldstone model. Kappa symmetry, from this point of view, allows half of the supersymmetries to be realized linearly. This fact has also been related [30] to the BPS property of fundamental strings. Let us mention finally that there is another framework, *doubly supersymmetric*, in which kappa-symmetry appears as a consequence of a local fermionic invariance of the world-volume [44, 45].

In another important work, Hughes, Liu and Polchinski [46] first generalized this set up for 3-branes in $d = 6$ dimensions.

In order to construct a κ-symmetric Green-Schwarz action [47] for the string, moving on Minkowski space, we start, following [42], from the supersymmetric 1-forms

$$\begin{cases} \omega^\mu = dX^\mu \;-\; i\bar{\theta}^B\Gamma^\mu d\theta^B\,, \\ d\theta^A\,, \end{cases} \tag{2.76}$$

where $\theta^A(\xi^i)$, the ξ^i are the coordinates on the worldsheet, are two $d = 10$ MW fermions and, at the same time, world-sheet scalars.

The "kinetic energy" part of the GS action is given by

$$\begin{cases} \mathcal{L}_1 \;=\; -\dfrac{1}{2}\sqrt{|h|}h^{ij}\omega_i^\mu\omega_j^\nu\eta_{\mu\nu}\,, \\ \omega_i^\mu \;=\; \partial_i X^\mu \;-\; i\bar{\theta}^A\Gamma^\mu\partial_i\theta^A\,. \end{cases} \tag{2.77}$$

As emphasized before, it is easy to check that this part by itself has supersymmetry realized in a nonlinear way. This fact can be interpreted [30] as an indication that, generically, an extended object will break all supersymmetries. It turns out that there is, in addition, a closed (actually exact), Lorentz and SUSY invariant three-form in superspace, namely

$$\Omega_3 \;=\; i\left(\omega^\mu \wedge d\bar{\theta}^1 \Gamma_\mu \wedge d\theta^1 \;-\; \omega^\mu \wedge d\bar{\theta}^2 \Gamma_\mu \wedge d\theta^2\right), \tag{2.78}$$

with $\Omega_3 = d\Omega_2$ and

$$\Omega_2 \;=\; -idX^\mu \wedge \left[\bar{\theta}^1 \Gamma_\mu d\theta^1 \;-\; \bar{\theta}^2 \Gamma_\mu d\theta^2\right] \;+\; \bar{\theta}^1 \Gamma^\mu d\theta^1 \wedge \bar{\theta}^2 \Gamma_\mu d\theta^2 \,. \tag{2.79}$$

Ω_2 is SUSY invariant up to a total derivative. The GS action is just

$$\mathcal{S}_{GS} \;=\; \int \left[\mathcal{L}_1 \;+\; \Omega_2\right], \tag{2.80}$$

and can be shown to be invariant under the transformations

$$\begin{cases} \delta\theta^A & = \epsilon^A \,, \\ \delta X^\mu & = i\bar{\epsilon}^A \Gamma^\mu \theta^A \,. \end{cases} \tag{2.81}$$

Now some of the supersymmetries are realized in a linear way, which physically means that the extended object is BPS, and thus preserves half of the supersymmetries.

Let us now turn our attention to the supermembrane. In [48] the following GS-type action was proposed for a supermembrane coupled to d=11 supergravity

$$S = \int d^3\xi \left\{ \frac{1}{2}\sqrt{-g}\, g^{ij} E_i^A E_j^B \eta_{AB} + \epsilon^{ijk} E_i^A E_j^B E_k^C B_{CBA} - \frac{1}{2}\sqrt{-g} \right\} . \tag{2.82}$$

Here the ξ^i (i=0,1,2) label the coordinates of the bosonic world-volume, and the (target-space) superspace coordinates are denoted by $Z^M(\xi)$. The action just represents the embedding of the three dimensional world-volume of the membrane, in eleven dimensional superspace. Lowercase latin indices will denote vectorial quantities; Lower case greek indices will denote spinorial quantities. Capital indices will include both types. Frame indices are denoted by the first letters of the alphabet, whereas curved indices will be denoted by the middle letters. On the other hand, $E_i^A \equiv E^A{}_B \partial_i Z^B$, and the super-three form B is the one needed for the superspace description of $d = 11$ supergravity.

Bergshoeff, Sezgin and Townsend imposed invariance under κ-symmetry, that is, under

$$\delta E^a = 0 \,, \tag{2.83}$$

$$\delta E^\alpha = (1+\Gamma)^\alpha{}_\beta \kappa^\beta \,, \tag{2.84}$$

$$\delta g_{ij} = 2X_{ij} - g_{ij} X^k_k \,, \tag{2.85}$$

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where $\kappa^\alpha(\xi)$ is a Majorana spinor and a world-volume scalar, $\delta E^A \equiv \delta Z^B E_B^A$ and $\Gamma^\alpha_\beta \equiv \frac{1}{6}\sqrt{g}\epsilon^{ijk}E_i^a E_j^b E_k^c(\gamma_{abc})^\alpha{}_\beta$. X_{ij} is a function of the E_i^A which should be determined by demanding invariance of the action. They found that for consistency they had to impose the constraints

$$H_{\alpha\beta\gamma\delta} = H_{\alpha\beta\gamma d} = H_{\alpha abc} = \eta_{c(a}T^c_{b)\alpha} = 0\,, \tag{2.86}$$

$$T^a{}_{\alpha\beta} = (\Gamma^a)_{\alpha\beta}\,, \tag{2.87}$$

$$H_{\alpha\beta ab} = -\frac{1}{6}(\Gamma_{ab})_{\alpha\beta}\,, \tag{2.88}$$

where $H_{\ldots}$ and $T^{\cdot}{}_{..}$ are the components of the super-fieldstrength of B and the supertorsion respectively. It is a remarkable fact that these constraints (as well as the Bianchi identities) are solved by the superspace constraints of $d = 11$ supergravity as given by Cremmer and Ferrara in [49].

This is the first of our encounters with some deep relationship between world-volume and spacetime properties: by demanding κ-symmetry (a world-volume property), we have obtained some spacetime equations of motion which must be satisfied for the world-volume symmetry to be possible at all.

2.7.2 Conformal invariance

There seems to be something special about the case $p = 1$ (*strings*). We have then, as we shall see, some extra symmetry, *conformal invariance*, which allows for the construction of a seemingly consistent perturbation theory.

There are no strings in $d = 11$: from our present point of view the most natural way of introducing them is through *double dimensional reduction* of the 11-dimensional membrane (*M-2-brane*). If we start with the (bosonic part of the) previous action of [48] for the latter, namely:

$$S_3 = T_3\int d^3\xi\,\Big[\frac{1}{2}\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^M\partial_j X^N G^{(11)}_{MN}(X) - \\ -\frac{1}{2}\sqrt{-\gamma} + \frac{1}{3!}\epsilon^{ijk}\partial_i X^M\partial_j X^N\partial_k X^P A^{(11)}_{MNP}(X)\Big]\,, \tag{2.89}$$

and follow [50], in assuming that there are isometries both in the spacetime generated by ∂_Y, and in the world-volume as well, generated by ∂_ρ. (We label spacetime coordinates as $X^M = (X^\mu, Y)$ where $M = 0,\ldots,10$ and $\mu = 0,\ldots,9$; World-volume coordinates as $\xi^i = (\xi^a,\rho)$, where i = 0, 1, 2 ; a = 0, 1). We now identify the two ignorable coordinates (*static gauge*)

$$\rho = Y\,, \tag{2.90}$$

and perform stardard Kaluza-Klein reduction, namely

$$G^{(11)}_{\mu\nu} = e^{-\frac{2\phi}{3}}(G^{(10)}_{\mu\nu} + e^{2\phi}A_\mu A_\nu)\,, \qquad G^{(11)}_{\mu Y} = -e^{\frac{4\phi}{3}}A_\mu\,, \qquad G^{(11)}_{YY} = e^{\frac{4\phi}{3}}\,, \tag{2.91}$$

and for the three-form

$$A^{(11)}_{\mu\nu\kappa} = A^{(10)}_{\mu\nu\kappa}\,, \qquad A^{(11)}_{\mu\nu Y} = B^{(10)}_{\mu\nu}\,. \tag{2.92}$$

This then leads to the action for a ten-dimensional string, namely

$$S_2 = T_2 \int d^2\xi \left[\frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G^{(10)}_{\mu\nu}(X) + \frac{1}{2}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B^{(10)}_{\mu\nu}(X)\right]. \tag{2.93}$$

This is a remarkable action,[6] which enjoys both two-dimensional reparametrization invariance and ten-dimensional invariance under the isometry group of the target-manifold (including the appropriate torsion) and, most importantly, under Weyl transformations

$$\gamma_{ab} \to e^{\psi(\xi)}\gamma_{ab}\,. \tag{2.94}$$

On the other hand, a well-known mathematical theorem ensures that, locally, any two-dimensional (Euclidean) metric can be put in the form

$$\gamma_{ab} = e^{\sigma}\delta_{ab}\,. \tag{2.95}$$

Owing to Weyl invariance, the trace of the energy-momentum tensor vanishes

$$0 = T^a_a \equiv \gamma^{ab}\frac{2}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma^{ab}}S[\gamma] = \frac{\delta}{\delta\sigma}S[\gamma]\,. \tag{2.96}$$

This, in its turn, means that the two-dimensional action is conformally invariant in flat space: that is, invariant under conformal Killing transformations $\delta\xi^a = k^a$, with

$$\partial_a k_b + \partial_b k_a = \delta_{ab}\,\partial_c k^c\,, \tag{2.97}$$

with reduces to

$$\partial_1 k_1 = \partial_2 k_2\,, \qquad \partial_1 k_2 = -\partial_2 k_1\,, \tag{2.98}$$

which in turn implies

$$\Box k_a = 0\,, \tag{2.99}$$

an infinite group; In terms of the natural coordinates $\xi^{\pm} \equiv \xi^0 \pm \xi^1$ the general two-dimensional conformal transformation is:

$$\delta\xi_a = f_a(\xi^+) + g_a(\xi^-)\,, \tag{2.100}$$

with arbitrary functions f_a and g_a. This infinite conformal symmetry is the root of many aspects of the physics of strings.

[6] Were we to reduce the kappa-symmetric supermembrane action we would have found the kappa-symmetric Green-Schwarz string action in d=10.

2.8 The string scale and the string coupling constant

If the radius of the eleventh dimension is R, and we denote the M-2-brane tension by $T_3 \equiv l_{11}^{-3}$, the string tension (traditionally denoted by α') will be given by $T_2 = \frac{R}{l_{11}^3} \equiv \frac{1}{l_s^2} \equiv \frac{1}{\alpha'}$. This gives the string length as

$$l_s = \frac{l_{11}^{3/2}}{R^{1/2}} \,. \tag{2.101}$$

The mass of the first Kaluza-Klein excitation with one unit of momentum in the eleventh direction is $M(KK) \equiv R^{-1}$. As we shall see later on, this state is interpreted, from the 10-dimensional point of view, as a D0-brane, and its mass could serve as a *definition* of the *string coupling constant*, $M(D0) \equiv \frac{1}{g_s l_s}$. Equating the two expressions gives

$$g_s = \frac{R}{l_s} = \left(\frac{R}{l_{11}}\right)^{3/2} \,. \tag{2.102}$$

This formula is very intriguing, because it clearly suggests that the string will only live in 10 dimensions as long as the coupling is small. The historical way in which Witten [51] arrived to this result was exactly the opposite, by realizing that the mass of a D0 brane (in 10 dimensions) goes to zero at strong coupling, and interpreting this fact as the opening of a new dimension. Although some partial evidence exists on how the full O(1,10) can be implemented in the theory (as opposed to the O(1,9) of ten-dimensional physics), there is no clear understanding about the rôle of conformal invariance (which is equivalent to BRST invariance, and selects the critical dimension) in eleven dimensional physics. We shall raise again some of these points in the section devoted to the strong coupling limit.

The radius could also be eliminated, yielding the beautiful formula

$$g_s = \left(\frac{l_{11}}{l_s}\right)^3 \,. \tag{2.103}$$

3. Conformal field theory

Starting from the classic work of Belavin, Polyakov and Zamolodchikov [52] the study of two-dimensional conformally invariant quantum field theories (CFT) has developed into a field of study on its own (see for example the textbooks [54, 55]), with applications in statistical physics [53]. From the point of view of strings, the imbeddings $x^\mu(\xi)$ are to be considered as two-dimensional quantum fields.

3.1 Primary fields and operator product expansions

In *any* d, the group of (euclidean) conformal transformations, $C(d) \sim O(1, d+1)$, is defined by all transformations $x^\mu \to x'^\mu(x)$ such that

$$\delta_{\mu\nu} dx'^\mu dx'^\nu \;=\; \Omega^{-2}(x)\, \delta_{\mu\nu} dx^\mu dx^\nu \,. \tag{3.1}$$

Given a conformal transformation we may define a corresponding local transformation by

$$\mathcal{R}_{\mu\alpha} \equiv \Omega(x) \frac{\partial x'_\mu}{\partial x_\alpha}, \qquad \mathcal{R}_{\mu\alpha}\mathcal{R}_{\nu\alpha} = \delta_{\mu\nu}. \tag{3.2}$$

A *quasi-primary* field [56] $\mathcal{O}^i$, (where i denotes the components in some space on which some representation of $\mathrm{O}(d)$ acts) , is defined to transform as

$$\mathcal{O}'^i(x') = \Omega^h(x) D^i{}_j(\mathcal{R})\, \mathcal{O}^j(x), \tag{3.3}$$

where h is called the scale dimension of the field. A quasi-primary field is called a *primary* field if it transforms as a scalar under the action of $\mathrm{O}(d)$.

We have previously seen that in $d = 2$ conformal transformations are of the type

$$\delta\xi_a = f_a(\xi^+) + g_a(\xi^-). \tag{3.4}$$

We will frequently be interested in CFT on the *cylinder*, $S^1 \times \mathbb{R}$, where the two-dimensional Lorentzian coordinates (τ, σ) are such that $\sigma = \sigma + 2\pi$. Performing a two-dimensional Wick rotation $\tau \to -i\tau$, $\xi^\pm \equiv \tau \pm \sigma \to -i(\tau \pm i\sigma)$, the coordinate $z \equiv \tau - i\sigma$ describes the (Wick rotated) cylinder.

One can now perform a conformal transformations (physics should be insensitive to this) to the Riemann sphere (the extended complex plane), $z \to e^z$. Quite frequently, coordinates on the Riemann sphere will also be denoted by z. Translations in $\tau \equiv \xi^0$, $\delta\tau = \epsilon$, map on the complex plane into $|z| \to e^\epsilon |z|$. Regular time evolution in τ on the cylinder then maps onto radial evolution from the origin of the complex plane (corresponding to the point $\tau = -\infty$ on the cylinder). In order to emphasize this, quantization on the complex plane is sometimes refered to as *radial quantization*. The energy momentum tensor T_{ab} represents the response of the action to a variation of the two-dimensional metric. Given any Killing vector field, k_a, the currents $T_{ab}k^b$ are conserved. This includes in particular all conformal transformations.

For open strings (with $0 \leq \sigma \leq \pi$) this conformal transformation maps the strip (σ, τ) into the upper half of the complex plane. A further conformal transformation could be used to map it into the unit disc; for example

$$z \to \frac{z - i}{z + i}, \tag{3.5}$$

maps the origin ($\tau = -\infty$) to the point -1 and semi-circles around the origin into arcs corresponding to circles centered in the real axis. The region $\tau \to \infty$ is mapped onto the single point $+1$.

In complex coordinates the $2d$ metric locally reads $ds^2 = dzd\bar{z}$, and one can see that the tracelesness and the conservation of the stress tensor read

$$T^\alpha{}_\alpha = T^z{}_z + T^{\bar{z}}{}_{\bar{z}} = 2T_{z\bar{z}} = 0, \tag{3.6}$$

$$\partial_\mu T^\mu{}_z = \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0. \tag{3.7}$$

The last equation then means that

$$\partial_{\bar{z}} T_{zz} \equiv \bar{\partial} T = 0 \,, \tag{3.8}$$

where, from now on, we will display the holomorphic part only.

The action for a massless scalar field, such as any of the d imbedding functions $x^{\mu}(z)$, is given by[7]

$$\mathcal{S} = \frac{1}{2\pi} \int d^2 z \, \partial\phi \bar{\partial}\phi = \frac{1}{4\pi i} \int d^2 x \, \partial_{\mu}\phi \partial^{\mu}\phi \,, \tag{3.9}$$

and the equation of motion reads

$$\partial\bar{\partial}\phi = 0 \,. \tag{3.10}$$

This means that *on-shell*

$$\phi(z,\bar{z}) = \frac{1}{2} \left(\phi(z) + \bar{\phi}(\bar{z}) \right) . \tag{3.11}$$

The propagator must solve the differential equation

$$\partial\bar{\partial} \langle T\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle = -2\pi\delta^{(2)}(z_1 - z_2, \bar{z}_1 - \bar{z}_2) \,, \tag{3.12}$$

which after using the formula

$$\bar{\partial}\frac{1}{z} = \pi\delta^{(2)}(z,\bar{z}) \,, \tag{3.13}$$

results in

$$\langle T\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle = -log(|z_1 - z_2|^2) \,. \tag{3.14}$$

It is customary to omit corresponding expressions for the anti-holomorphic part, and write down explicitely the holomorphic part only

$$\langle T\phi(z_1)\phi(z_2)\rangle = -log(z_1 - z_2) \,. \tag{3.15}$$

Wick's theorem ensures that the T-product is expressible as

$$T\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2) =: \phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2) : + \phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2) \,. \tag{3.16}$$

Clearly by construction

$$\langle : \phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2) : \rangle = 0 \,, \tag{3.17}$$

and the normal-ordered product obeys the classical equation of motion, without source terms

$$\partial\bar{\partial} : \phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2) := 0 \,. \tag{3.18}$$

[7]The transformation between the coordinates is given by $z = x + iy$, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$.

This means that there is a naïve Operator Product Expansion (OPE), given simply by the Taylor expansion whose holomorphic part is

$$\phi(z)\phi(w) = -log(z-w) + \, :\phi\phi:(w): + (z-w):\phi\partial\phi:(w) + \cdots \tag{3.19}$$

Contractions then represent the singular part of the OPE. Correlators in free theories (such as the most interesting examples in String Theory) are given by the general form of Wick's theorem

$$\langle :A_1(z_1)\cdots A_n(z_n):\cdots :D_1(w_1)\cdots D_m(w_m):\rangle \tag{3.20}$$

is given by the sum of all possible pairings, excluding those corresponding to operators inside the same normal ordering.

Using the above rules we obtain the OPE

$$\partial\phi(z)\partial\phi(w) \sim -\frac{1}{(z-w)^2}\,. \tag{3.21}$$

The energy momentum tensor corresponding to a scalar field coupled minimally to the two-dimensional metric is given by

$$T(z) = -\frac{1}{2}:(\partial\phi)^2:(z)\,. \tag{3.22}$$

Given a (conformal) Killing, k^a, there is an associated conserved current, given by $j_a \equiv T_{ab}k^b$. Its conserved charge is given on the cylinder by

$$Q(k) \equiv \int_{\tau=cons.} T_{0b}k^b d\sigma\,. \tag{3.23}$$

A conformal transformation $z \to z + \xi(z)$ is associated on the plane to the charge

$$Q(\xi) \equiv \oint \frac{dz}{2\pi i}T(z)\xi(z)\,. \tag{3.24}$$

The corresponding transformation of a field $\partial\phi$ will be given by

$$\delta\left[\partial\phi(z)\right] \equiv \langle[Q(\xi),\partial\phi(z)]\rangle\,. \tag{3.25}$$

The fact that path integrals in the plane are automatically radially ordered allows for a simple representation of correlators in terms of Cauchy integrals

$$\begin{aligned} &= \int \mathcal{D}\phi\, e^{-\mathcal{S}}\Big[Q_\xi(|z|+\epsilon)\partial\phi(z) - \partial\phi(z)Q_\xi(|z|-\epsilon)\Big] \\ &= \oint_{C_z}\frac{dw}{2\pi i}\xi(w)\langle T(w)\partial\phi(z)\rangle\,, \end{aligned} \tag{3.26}$$

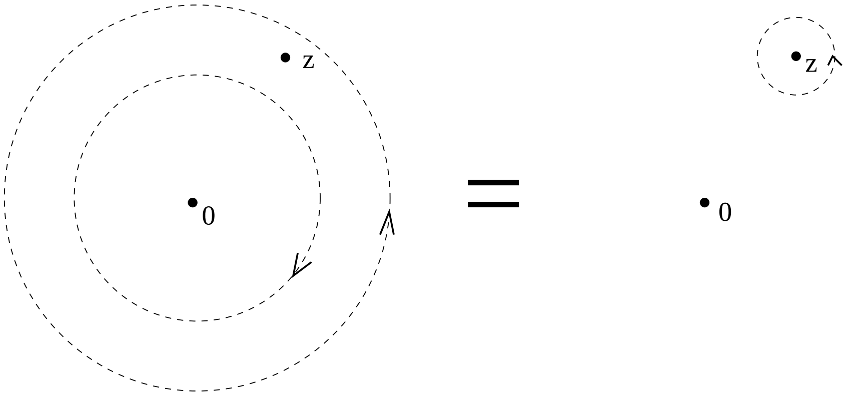

Figure 2: The needed contour deformation.

where the two contours are deformed as in fig. 2 into a single one C_z around the privileged point z, using the fact that the conserved charge, $Q(\xi)(w)$ is independent of w, which is implemented mathematically by the fact that we can deform the contour of definition of Q as long as we do not meet any singularities, which is possible only at z. Since we are dealing with a free theory, we can use Wick's theorem to evaluate the last v.e.v., i.e.

$$\begin{aligned}
&= -\frac{1}{2}\oint \frac{dw}{2\pi i}\,\langle :\partial\phi\partial\phi : (w)\partial\phi(z)\rangle \xi(w) \\
&= \oint \frac{dw}{2\pi i}\,\frac{1}{(w-z)^2}\partial\phi(w)\xi(w) \\
&= \oint \frac{dw}{2\pi i}\left[\frac{\partial\phi(z)}{(z-w)^2} + \frac{\partial^2\phi(z)}{w-z} + \cdots\right]\left[\xi(z) + \partial\xi(z)\,(w-z) + \cdots\right] \\
&= \partial\phi(z)\partial\xi(z) + \partial^2\phi(z)\xi(z)\,.
\end{aligned} \tag{3.27}$$

In general, for a field, $\phi^{(\lambda)}$, of arbitrary scaling dimension λ, we would have

$$T(z)\phi^{(\lambda)}(w) = \frac{\lambda\phi^{(\lambda)}(w)}{(z-w)^2} + \frac{\partial\phi^{(\lambda)}(w)}{z-w}\,. \tag{3.28}$$

This is telling us that the scaling dimension of $\partial\phi$ is one,

$$h(\partial\phi) = 1\,. \tag{3.29}$$

A conformal, dimension 1, field can be expanded in Fourier modes as

$$\partial\phi(z) = \sum \frac{\alpha_m}{z^{m+1}}\,. \tag{3.30}$$

Note that we have written z^{m+1} instead of z^m; This is an effect of the conformal mapping from the cylinder to the plane.

Other important primary fields associated to a scalar field are the vertex operators $V_\alpha(z) =: e^{i\alpha\phi(z)} :$. It is a simple exercise to show that

$$T(z)V_\alpha(w) = \frac{\alpha^2 V_\alpha(w)}{2(z-w)^2} + \frac{\partial V_\alpha(w)}{z-w}\,, \tag{3.31}$$

$$\langle V_\alpha(z)V_{-\alpha}(w)\rangle = (z-w)^{-\alpha^2}\,, \tag{3.32}$$

$$: e^{i\alpha\phi(z)} :: e^{i\beta\phi(w)} : = (z-w)^{\alpha\beta} : e^{i\alpha\phi(z)+i\beta\phi(w)} :\,, \tag{3.33}$$

meaning that the scaling dimension of a vertex operator V_α is $h\,(V_\alpha) = \alpha^2/2$.

Note that derivatives of primary fields are not primary fields (BPZ calls them secondary fields) [52].

In CFT there is a natural mapping from operators to states, given by the path integral with an operator insertion, in terms of boundary values of this operator on the unit circle.

$$\Theta \longrightarrow \int_{\phi(|z|=1)=\phi_B} \mathcal{D}\phi\,\Theta\,[\phi]\,e^{-S[\phi]} = \Psi\,(\phi_B) \equiv |\Theta\rangle. \tag{3.34}$$

The in-vacuum (the state at $\tau = -\infty$, that is $z = 0$ on the plane), corresponds to the unit operator.

In order to study scattering states, let $\mid 0\rangle$ be an asymptotic, $\tau \to -\infty$, state without any insertion. Then if the action of the operator at the origin is to be well defined,

$$\phi(z) \mid 0\rangle = \sum \phi_n z^{-n-h} \mid 0\rangle\,, \tag{3.35}$$

it is necessary that

$$\phi_n \mid 0\rangle = 0\,, \tag{3.36}$$

for $n + h > 0$.

Let us next consider states constructed out of vertex operators of the form $e^{ip_\mu X^\mu(0)} \mid 0\rangle$. They represent the asymptotic state of a ground-state string at momentum p^μ. It should be easy to prove that

$$\alpha^\mu_m e^{ip_\mu X^\mu(0)} \mid 0\rangle = \delta_{n,0} p^\mu\, e^{ip_\mu X^\mu(0)} \mid 0\rangle\,. \tag{3.37}$$

Other excited states are represented by composite operators of the type

$$\partial X^\mu e^{ip_\mu X^\mu}\,,\ \partial X^\mu \partial X^\nu e^{ip_\mu X^\mu}\,,\ \text{et cetera}\ldots \tag{3.38}$$

where the p^2 has to be chosen, such that the conformal dimension of the operator equals 1 (see section 4.4).

Let us now consider the $\tau \to +\infty$ behaviour: expand an arbitrary dimension h field ϕ as $\phi = \sum_n \phi_n z^{-n-h}$ and have a look at the *out* state defined by

$$\langle\phi\mid = \lim_{w\to\infty}\langle 0 \mid \phi(w) = \lim_{z\to 0}\langle 0 \mid \phi\left(\frac{1}{z}\right) z^{-2h} \equiv \langle 0 \mid \phi^+\,. \tag{3.39}$$

The BPZ adjoint is then defined by

$$\phi^{\dagger} \equiv \sum_n \phi_n z^{n+h} z^{-2h} = \sum_n \phi_n^{\dagger} z^{-n-h}, \tag{3.40}$$

Summarizing, then, the *in* and *out* vacua obey

$$\phi_n \mid 0\rangle = 0\,(n > -h), \qquad \langle 0 \mid \phi_n = 0\,(n < h). \tag{3.41}$$

3.2 The Virasoro algebra

Classically conformally invariant theories do not, in general, preserve this property at the quantum level, because of the well-known *trace anomaly* [57].

Given any Conformal Field Theory with conformal anomaly coefficient c, the manifestation of this anomaly can be altered somewhat by local counterterms. There is a definition of the energy-momentum tensor such that it is conserved, but there is a trace anomaly *sensu stricto*

$$\begin{aligned} T^{\mu}_{(trace)\mu} &= -\frac{c}{6} R(g), \\ \nabla_{\mu} T^{\mu\nu}_{(trace)} &= 0. \end{aligned} \tag{3.42}$$

Another definition in such a way that it is traceless, but not conserved, so that there is now a gravitational anomaly

$$\begin{aligned} T^{\mu}_{(grav)\mu} &= 0, \\ \nabla_{\mu} T^{\mu\nu}_{(grav)} &= \frac{c}{12} \nabla^{\nu} R(g). \end{aligned} \tag{3.43}$$

Actually this is the definition such that the Fourier modes of $T^{(grav)}_{\mu\nu}$, after the addition of the improvement terms [17]

$$-\frac{c}{6\pi}\left(2\partial^2\phi - (\partial\phi)^2\right), \tag{3.44}$$

where ϕ is the Liouville field to be introduced in section 4.1, span a Virasoro algebra.

The Virasoro algebra in OPE notation reads

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{3.45}$$

It is an easy exercise to show that the energy momentum of the scalar field obeys the above equation with $c = 1$.

The most important property of the mapping from the cylinder to the plane is

$$T_{cylinder}(z') = z^2 T_{plane}(z) - \frac{c}{24}, \tag{3.46}$$

which one can deduce using the above rules.

Expanding $T(z)$ in Fourier series

$$T(z) \;=\; \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,, \tag{3.47}$$

so that

$$L_n \;=\; \oint \frac{dz}{2\pi i}\, z^{n+1}T(z)\,, \tag{3.48}$$

one can compute the commutators using OPEs, leading to

$$[L_n, L_m] = (n-m)L_{n+m} \;+\; \frac{c}{12}n(n+1)(n-1)\delta_{m,-n}\,. \tag{3.49}$$

Direct inspection shows that $\left\{L_0, L_{\pm 1}, \bar{L}_0, \bar{L}_{\pm 1}\right\}$ generate the algebra $\mathrm{SL}(2,\mathbb{C})$, and that L_0 generates dilatations $\xi(z) = z$.

It is also possible to show, using the definition of conformal weight, and expanding an arbitrary field $\phi^{(\lambda)}$ as $\phi^{(\lambda)}(z) = \sum_n \phi_n^{(\lambda)} z^{-n-\lambda}$ that

$$\left[L_m, \phi_n^{(\lambda)}\right] \;=\; \left[(\lambda-1)m - n\right]\phi_{n+m}^{(\lambda)}\,. \tag{3.50}$$

3.3 Non-minimal coupling and background charge

Although the *minimal* coupling of a scalar field to the two-dimensional metric consists simply in writing covariant derivatives instead of ordinary ones, there are more complicated (*non-minimal*) possibilities. One of the most interesting involves a direct coupling to the two-dimensional scalar curvature [58]

$$\mathcal{S}_Q \;=\; \frac{1}{8\pi}\int d^2z\,\left[\partial\phi\bar{\partial}\phi \;-\; 2Q\sqrt{g}R^{(2)}\phi\right]\,, \tag{3.51}$$

for some Q.

It is possible to show that the holomorphic stress tensor reads

$$T(z) \;=\; -\frac{1}{2} :\partial\phi\partial\phi:(z) \;+\; Q\partial^2\phi(z)\,. \tag{3.52}$$

Introducing the, formerly conserved, current $j = -\partial\phi$, and using the fact that the propagator for ϕ does not change, we can calculate

$$T(z)j(w) \;=\; \frac{-2Q}{(z-w)^3} \;+\; \frac{j(w)}{(z-w)^2} \;+\; \frac{\partial j(w)}{z-w}\,, \tag{3.53}$$

$$j(z)e^{q\phi(w)} \;=\; \frac{q}{z-w}\,e^{q\phi(w)}\,, \tag{3.54}$$

$$T(z)e^{q\phi(w)} \;=\; -\frac{q(q+2Q)}{2(z-w)^2}e^{q\phi(w)} \;+\; \frac{\partial e^{q\phi(w)}}{z-w}\,, \tag{3.55}$$

as well as

$$T(z)T(w) = \frac{1+12Q^2}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{3.56}$$

showing that T generates a Virasoro algebra with $c = 1 + 12Q^2$. Note that the current behaves as an anomalous primary field, obviously due to the non-minimal coupling. The equation of motion, written in terms of the current j, already entails the occurrence of the anomaly, e.g.

$$\bar{\partial} j = Q\sqrt{g}R^{(2)} . \tag{3.57}$$

Using the anomalous transformation law for j one can show that under the transformation $z \to w = z^{-1}$, we have

$$\partial\phi(z) = -w^2\partial\phi(w) + 2Q\,w . \tag{3.58}$$

Using this, we get on the Riemann sphere, putting $Q = -i\alpha_0$

$$Q(z=0) = \oint \frac{dz}{2\pi i}\partial\phi(z) = \oint \frac{dw}{2\pi i}\partial\phi(w) - \oint \frac{dw}{2\pi i}\frac{2Q}{w} = Q(z=\infty) - 2Q . \tag{3.59}$$

If we define $\mid \alpha\rangle = V_\alpha(0) \mid 0\rangle$, then

$$\langle \alpha \mid \equiv \lim_{z\to\infty} \langle 0 \mid V_\alpha(z)\, z^{2h_\alpha} , \tag{3.60}$$

with $2h_\alpha = \alpha(\alpha - 2\alpha_0)$. We can calculate

$$\langle \alpha \mid \beta\rangle = \langle 0 \mid V_\alpha(w^{-1})\, w^{-\alpha(\alpha-2\alpha_0)} V_\beta(z) \mid 0\rangle \sim w^{-\alpha(\alpha-2\alpha_0)}\, (w^{-1} - z)^{\alpha\cdot\beta} , \tag{3.61}$$

which has a smooth and non-vanishing limit for $w \downarrow 0$ iff $\beta = 2\alpha_0 - \alpha$.

3.4 (b, c) systems and bosonization

It is very interesting to consider a system of (anti-)commuting analytical fields of conformal weights j and $1-j$, usually called b_j and c_{1-j}. These systems appear, in particular, when fixing the (super)conformal gauge invariance. It is possible to consider both cases simultaneously by introducing a parameter ϵ, valued $+1$ in the anticommuting case, and -1 in the commuting case. The action for these fields, first order in derivatives, is

$$S \equiv \frac{1}{2\pi}\int b\bar{\partial}c \tag{3.62}$$

It follows that

$$\begin{cases} b(z)c(w) = \dfrac{\epsilon}{z-w} , \\ T(z) = -j\ : b\partial c :\ + (1-j) : \partial b \cdot c : \ . \end{cases} \tag{3.63}$$

(Please note that $c(z)b(w) = \frac{1}{z-w}$).

The above stress tensor satisfies

$$T(z)T(w) = -\epsilon\frac{6j^2 - 6j + 1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{(\partial T)(w)}{z-w} . \tag{3.64}$$

This means that the conformal anomaly for the (b,c) system reads

$$c = -2\epsilon\,[6j(j-1)+1]\,, \tag{3.65}$$

The physical (b,c) systems needed in the quantization of superstrings are an anticommuting system of ghosts, denoted (b,c), due to the gauge fixing of the diffeomorphisms, and a commuting system, denoted (β,γ), due to the gauge fixing of local supersymmetry. Their characteristics are

$$\begin{array}{lll} (b,c): & c=-26\,, & (j=2)\,, \\ (\beta,\gamma): & c=11\,, & (j=\frac{3}{2})\,. \end{array} \tag{3.66}$$

3.4.1 Bosonization

In two dimensions there is no essential difference between bosons and fermions, and, in particular, it is possible to bosonize fermionic expressions. All our relationships are to be understood, as usual, valid inside correlators only.

A free boson is a $c=1$ CFT and as such one can show that it is equivalent to two minimal spinors. The operator correspondence is generated by

$$\psi_1 + i\psi_2 = \sqrt{2}\,e^{i\phi}\,. \tag{3.67}$$

This is the simplest instance of bosonization: the starting point of the whole construction.

The (b,c) system on the other hand, is equivalent to a non-minimally coupled boson, denoted σ. Identification of the conformal anomaly $c_{bc} = -26$ leads to, using the formula for a non minimally coupled scalar $c = 1+12Q^2$, a value for the background charge of $Q=-i\frac{3}{2}$. Using this fact, one can show that

$$T(z)e^{i\alpha\sigma} = \left(\frac{\alpha^2}{2} - \frac{3\alpha}{2}\right)\frac{1}{(z-w)^2}\,e^{i\alpha\sigma} + \cdots \tag{3.68}$$

This suggests that the correct mapping of fields is given by

$$\begin{array}{ll} b(z) = e^{-i\sigma}: & (j=2)\,, \\ c(z) = e^{i\sigma}: & (j=-1)\,. \end{array} \tag{3.69}$$

Although we are going to be quite schematic about it, it is also possible to bosonize the (already bosonic) (β,γ) system. Actually, we write the (β,γ) system as a $c=13$ non-minimally coupled boson ϕ with background charge $Q=1$, and another $j=0$ (b,c)-system with $\epsilon=+1$ (that is, anticommuting), the (ξ,η) system, carrying a conformal anomaly of $c_{(0,1)}=-2$. The total stress tensor reads

$$T(z) = -\frac{1}{2} : \partial\phi\partial\phi : + \partial^2\phi + : (\partial\xi)\,\eta :, \tag{3.70}$$

and the resulting central charge is $c = 13-2 = 11$ as it ought to be. The explicit bosonization rules are then

$$\beta = \partial\xi\,e^{-\phi}\,, \qquad \gamma = \eta\,e^{\phi}\,. \tag{3.71}$$

3.5 Current algebras and the Frenkel-Kač-Segal construction

There is a kind of non-Abelian generalization of the Virasoro algebra, called Kač-Moody algebras, and is associated to a Lie algebra $[T^a, T^b] = if^{abc}T^c$. From our point of view, they are characterized by the OPE

$$J^a(z)J^b(w) \;=\; \frac{k\delta^{ab}}{(z-w)^2} + if^{abc}\frac{J^c(w)}{z-w} + \cdots \tag{3.72}$$

where k is the so-called level ("central element") of the Kač-Moody algebra. The Sugawara construction [60, 61] of the stress tensor stands for, in case of simple Lie algebras,

$$T(z) \;=\; \frac{1}{2k+c_2}\sum_a \,:J^aJ^a:(z)\,, \tag{3.73}$$

where c_2 is the value of the quadratic Casimir in the adjoint representation, which for simply laced groups $(A_n, D_n, E_6, E_7, E_8)$ is given by:

$$c_2 = 2\left(\frac{\dim(G)}{\text{rank}} - 1\right). \tag{3.74}$$

Computing $T(z)T(w)$ yields:

$$c \;=\; \frac{2k\,\dim(G)}{2k+c_2}\,. \tag{3.75}$$

(This implies, in particular, that $c(\mathrm{SU}(2)_{k=1}) = 1$). The value of the conformal anomaly lies between the rank of the group (the minimal possible value) and its dimension.

The simplest physical representation of a KM algebra is through a system of $2N$ two-dimensional fermions, satisfying

$$\psi^\mu(z)\psi^\nu(w) \;=\; -\delta^{\mu\nu}\frac{1}{z-w}\,, \tag{3.76}$$

such that the currents, the $T^a_{\mu\nu}$ are the generators of $\mathrm{SO}(2N)$ in the vector representation,

$$j^a(z) \;=\; \frac{1}{2}:\psi^\mu T^a_{\mu\nu}\psi^\nu:(z)\,, \tag{3.77}$$

generate an $\mathrm{SO}(2N)_{k=1}$ current algebra, as can easily be checked using the above OPEs. We can relabel the indices, using $\mathrm{SU}(N) \subset \mathrm{SO}(2N)$,

$$\psi^{\pm a} \;=\; \frac{1}{\sqrt{2}}\left(\psi^a \pm \psi^{a+1}\right), \qquad\qquad a = 1,\ldots,N\,, \tag{3.78}$$

in such a way that

$$\psi^{+a}(z)\psi^{-b}(w) \;=\; -\frac{\delta^{ab}}{z-w}\,, \tag{3.79}$$

This system can be bosonized, i.e. written in terms of N bosonic fields ϕ_a by a technique very similar to the one used in the previous paragraph, i.e.

$$\psi^a \;=\; i\mathcal{C}(a)\, e^{i\alpha_a\cdot\phi}\,, \tag{3.80}$$

where $\alpha_a^i = \delta_a^i$ is a weight of the vector representation of O$(2N)$ and we are forced to introduced the quantities $\mathcal{C}$, called cocycles, which satisfy

$$\begin{cases} \mathcal{C}(a)\mathcal{C}(b) = -\mathcal{C}(b)\mathcal{C}(a)\,, & a \neq b \\ \mathcal{C}(a)\mathcal{C}(a) = 1\,, & \end{cases} \tag{3.81}$$

This immediatly yields an exceedingly useful representation of the currents in terms of the vertex operators associated to the scalar fields

$$\begin{array}{ll} j^{+ab} = \mathcal{C}(a)\mathcal{C}(b)e^{i\alpha_{ab}\cdot\phi}\,, & \alpha^i_{ab} = \delta^i_a + \delta^i_b\,, \\ j^{+a\bar{b}} = \mathcal{C}(a)\mathcal{C}(b)e^{i\alpha_{a\bar{b}}\cdot\phi}\,, & \alpha^i_{a\bar{b}} = \delta^i_a - \delta^i_b\,, \\ j^{a\bar{a}} \;\; = i\partial\phi^a\,, & \end{array} \tag{3.82}$$

On the plane the corresponding charges are defined through

$$M_{a\bar{a}} = \frac{1}{2\pi i}\oint j_{a\bar{a}}(z)\,, \tag{3.83}$$

This procedure is known as the Frenkel-Kač-Segal (FKS) construction, although in the particular case of SU(2) it was anticipated by Halpern [62].

It is plain that all the preceding can be generalized to an arbitrary representation. Actually, for an arbitrary weight α^i, we have

$$j_{a\bar{a}}(w)e^{i\alpha_s\cdot\phi(z)} \;=\; \frac{\alpha_s^a}{w-z}\cdot e^{i\alpha_s\cdot\phi(z)}\,, \tag{3.84}$$

as well as

$$\psi^a e^{i\alpha_s\cdot\phi} \;\sim\; (z-w)^{\alpha_s^a}\;:i\mathcal{C}(a)e^{i(\alpha_a+\alpha_s)\cdot\phi}:\;. \tag{3.85}$$

In the particular case when α_s is the weight vector corresponding to a spinor representation, i.e. $(\pm\frac{1}{2},\ldots,\pm\frac{1}{2})$, the preceding OPE has a characteristic square root singularity, and is then called a spin operator, because it transforms as an O$(2N)$ spinor.

This process is quite remarkable: starting with two-dimensional spinors, which are also spacetime vectors, we have constructed, by bosonization, and vertex operators, a set of spacetime fermions. To be specific

$$S_A(z) \;=\; \mathcal{C}(A)e^{i\alpha_A\cdot\phi(z)}\,, \tag{3.86}$$

and the cocycles can be chosen such that

$$j^{ab}(z)S_A(w) \;=\; \frac{1}{z-w}\left(\frac{1}{4}\gamma^{ab}\right)_{AB} S_B(w)\,. \tag{3.87}$$

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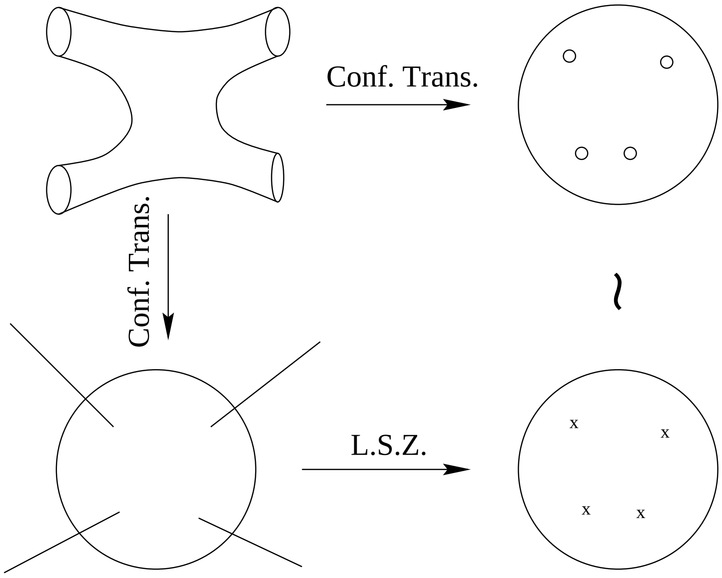

Figure 3: Picture showing the equivalence between the punctured sphere (punctures being the small circles) and the sphere with the insertion of the vertex operators, as depicted through the crosses.

4. Strings and perturbation theory

In string perturbation theory, Feynman diagrams are, as was to be suspected, thick versions of the usual Feynam diagrams.[8] One can then use conformal invariance (in the simplest closed string case) to map the diagram to a Riemann surface of genus g, with punctures on which we have to insert the string wavefunctions. This process is depicted in fig. 3 for a tree level scattering of four closed strings. If we then apply the Lehman-Symanzik-Zimmermann reduction-technique to this diagramm we get a compact surface but with the insertion of some local operators, called again vertex operators, bearing the quantum numbers of the external string states [14]. Strings have been studied from the vantage point of CFT in a classic paper by Friedan, Martinec and Shenker [63], where previous work is summarized. This is a highly technical subject and we can only give here a flavour of it. There is a very good review by E. and H. Verlinde [16].

4.1 The Liouville field: critical and non-critical strings

Polyakov [3] apparently was the first person to take seriously covariant path integral techniques to study string amplitudes. The zero point amplitude in the simplest closed string case is organized as

$$Z \equiv \sum_{g=0}^{\infty} \int_{\Sigma_g} \mathcal{D}\left[g_{ab}\right] e^{-S_{matt}(g)-\lambda\chi(\Sigma)}\,, \tag{4.1}$$

[8]One might say that the propagator is replaced by a cylinder.

where Σ_g is a two dimensional closed surface without boundary, with Euler characteristic $\chi(\Sigma_g) = 2g - 2$, g being the genus ($g = 0$ for the sphere, S^2, $g = 1$ for the torus T^2, etc., and

$$S_{matt}(g) \equiv \int_{\Sigma_g} d(vol) g_{ab} \partial^a \vec{X} \cdot \partial^b \vec{X} \,. \tag{4.2}$$

This action is classically invariant under both two-dimensional diffeomorphisms and Weyl rescalings. This means three parameters, which allows for a complete gauge fixing. In a somewhat symbolic notation, we can always reach the *conformal gauge* that is, we can write

$$g_{ab} = e^{2\phi} e^{\xi} \hat{g}_{ab}(\tau) \,, \tag{4.3}$$

where ϕ generates the Weyl transformation, ξ the diffeomorphism, and $\hat{g}_{ab}(\tau)$ is a fiducial metric on Σ_g. To be specific, locally the gauge

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \,, \tag{4.4}$$

can always be reached through diffeomorphisms, ($\delta g_{ab} \equiv \nabla_a \xi_b + \nabla_b \xi_a$), leaving $g_{z\bar{z}}$ to be traded for a Weyl rescaling. The path integral is then reduced to

$$Z \sim \int \mathcal{D}g \mathcal{D}X \, e^{-S_{matt}} \delta(g_{zz}) \delta(g_{\bar{z}\bar{z}}) \det \frac{\delta g_{zz}}{\delta \xi} \cdot \det \frac{\delta g_{\bar{z}\bar{z}}}{\delta \xi} \,. \tag{4.5}$$

The Faddeev-Popov determinants can as usual be represented by a ghost integral namely

$$e^{-W_{ghost}} \equiv \int \mathcal{D}c^z \mathcal{D}b_{zz} \mathcal{D}c^{\bar{z}} \mathcal{D}b_{\bar{z}\bar{z}} e^{-\frac{1}{\pi}\int d^2\sigma c^{\bar{z}} \nabla_z b_{\bar{z}\bar{z}} + c^z \nabla_{\bar{z}} b_{zz}} \,. \tag{4.6}$$

It is quite useful to keep in mind that the only non-vanishing Christoffel symbols for the metric

$$ds^2 = e^{2\phi} dz d\bar{z} \,, \tag{4.7}$$

are $\Gamma^z_{zz} = 2\partial\phi$ and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\bar{\partial}\phi$. This means that some covariant derivatives are just equivalent to the holomorphic derivative operator; that is

$$\nabla_{\bar{z}} t_{z_1 \ldots z_n} = \bar{\partial} t_{z_1 \ldots z_n} \,, \tag{4.8}$$

or

$$\nabla_{\bar{z}} t^{z_1 \ldots z_n} = \bar{\partial} t^{z_1 \ldots z_n} \,. \tag{4.9}$$

Other more complicated cases can be easily worked out. Finally, let us remark that the two-dimensional curvature is just

$$R^{(2)} = -2 e^{-2\phi} \partial\bar{\partial}\phi \,. \tag{4.10}$$

There are some small subtleties with this path integral. First of all, there could be ghost zero modes, i.e. solutions of the equation

$$\bar{\partial} c^z = 0 \,. \tag{4.11}$$

They are called *conformal Killing vectors*, and are related to diffeomorphisms which are equivalent to Weyl transformations. Their number is $C_0 = 3$ for the sphere, $C_1 = 1$ for the torus, and $C_g = 0$ for $g > 1$. To understand their meaning, let us note that under a reparametrization the metric changes as

$$\delta g_{\bar{z}\bar{z}} \equiv 2\nabla_{\bar{z}}\xi_{\bar{z}} = 2\nabla_{\bar{z}} g_{\bar{z}z}\xi^z = 2g_{\bar{z}z}\nabla_{\bar{z}}\xi^z \,. \tag{4.12}$$

This then shows that c's zero modes yield reparametrizations that are equivalent to a Weyl rescaling.

There could also be antighost zero modes (called *holomorphic quadratic differentials* by mathematicians), i.e. solutions of

$$\bar{\partial} b_{zz} = 0 \,. \tag{4.13}$$

To understand what this means, let us look at the action

$$\mathcal{S} = \int_\Sigma |\delta h_{zz} - \nabla_z \xi_z|^2 \neq 0 \,. \tag{4.14}$$

Minimizing this action leads to $\nabla_{\bar{z}} b_{zz} = 0$. This should hopefully make plausible the fact that antighost zero modes are related to deformations of the metric with non-vanishing action S as above. The physical meaning of them then lies in the fact that there are metrics on some Riemann surfaces not related by any gauge transformation (either Weyl or diffeomorphism) described by the so called *Teichmüller parameters.* There is none for the sphere, $B_0 = 0$, one for the torus, $B_1 = 1$, and $B_g = 3g - 3$ for $g > 1$. The *Beltrami differentials* μ, are defined from an infinitesimal variation in such a way that

$$\delta g_{zz} = \sum_i \delta\tau_i \mu_{izz} + \nabla_z \xi_z \tag{4.15}$$

The necessity to soak up zero modes means that it is neccessary to include a factor of

$$\prod_j | < \mu_j | b > |^2 \tag{4.16}$$

and to divide by the volume of the conformal Killing vectors, Vol(CKV) in lower genus.

At any rate we shall write the effective action as

$$W(g) \equiv W_{matt}(\hat{g}, \phi) + W_{ghost}(\hat{g}, \phi) \,. \tag{4.17}$$

Under a Weyl transformation $\delta\phi$ the conformal anomaly implies that

$$\delta W = \frac{26 - c}{12\pi} \int \sqrt{g} R(g) \delta\phi + \int \frac{\mu_0^2}{2\pi} \sqrt{g} \delta\phi \,, \tag{4.18}$$

which can also be written as

$$\frac{26-c}{12\pi}\int\sqrt{\hat{g}}(R(\hat{g})+\Delta_{\hat{g}}\phi)\delta\phi+\frac{\mu_0^2}{2\pi}\int e^{2\phi}\delta\phi\,. \tag{4.19}$$

This is easily integrated, yielding

$$W(\hat{g},\phi)=\frac{26-c}{12\pi}\int\sqrt{\hat{g}}\left(\frac{1}{2}\phi\Delta_{\hat{g}}\phi+R(\hat{g})\phi\right)+\int\frac{\mu_0^2}{4\pi}e^{2\phi}\,, \tag{4.20}$$

where μ_0^2, the world-sheet cosmological constant, comes from any explicit violation of conformal invariance in the trace of the energy-momentum tensor, $T^\alpha{}_\alpha=\frac{c}{6}R+\mu_0^2$.

This action was called by Polyakov the *Liouville action.* It clearly shows the difference between *critical strings* (which in the purely bosonic case we are considering would mean $c=26$), in which the Liouville action appears with zero coefficient, and *non-critical strings*, for all other values of c. In spite of a tremendous effort, in particular by the group at the Landau Institute, it is fair to say that our understanding of non-critical strings is still rather limited.

It is not difficult to rewrite the full Liouville action as a non-local R^2 term, which is sometimes useful

$$\begin{aligned}\mathcal{S}_L&=\int\left[\mu^2e^{2\phi}-4\phi\partial^2\phi\right]d^2\xi\\&=\int\mu^2e^{2\phi}-\int d^2\xi d^2\xi' e^{2\phi}e^{2\phi'}e^{-2\phi}e^{-2\phi'}4\partial^2\phi\partial^2\phi'\frac{e^{ik(\xi-\xi')}}{k^2}\frac{d^2k}{(2\pi)^2}\,.\end{aligned} \tag{4.21}$$

Using then the Fourier representation of the propagator

$$\Box^{-1}(\xi,\xi')\;=\;-\int\frac{e^{ik(\xi-\xi')}}{k^2}\frac{d^2k}{(2\pi)^2}\,, \tag{4.22}$$

we can rewrite the Liouville action as

$$\mathcal{S}_L\;=\;\int d^2\xi d^2\xi'\sqrt{g(\xi)}\sqrt{g(\xi')}R(\xi)\Box^{-1}(\xi,\xi')R(\xi')\;+\;\int\mu^2\sqrt{g}\,d^2\xi\,. \tag{4.23}$$

Once more, what is local and what is non-local depends on the variables used to describe the system.

One of the major difficulties in understanding Liouville comes from the fact that the *line element* implicit in the path integral measure $\mathcal{D}\phi$ is $||\delta\phi||^2\equiv\int e^{2\phi}\delta\phi^2$, which is not translationally invariant (as it would have been, had we used $\sqrt{\hat{g}}$ instead of $\sqrt{g}$). David, Distler and Kawai [64] made the assumption that all the difference can be summarized by a renormalization of all the parameters in the action, as well as a rescaling of the Liouville field itself

$$\mathcal{D}_g\phi=\mathcal{D}_{\hat{g}}\phi\cdot e^{-\frac{1}{2\pi}\int\left(\partial\phi\bar{\partial}\phi-\frac{1}{4}\tilde{Q}\sqrt{\hat{g}}R(\hat{g})\phi+\mu_1^2\sqrt{\hat{g}}e^{\alpha\phi}\right)}\,. \tag{4.24}$$

We can always fine-tune μ_0 so that $\mu_1 = 0$.

The original theory only depends on g, so we have a fake symmetry

$$\left.\begin{array}{l}\hat{g} \to e^{\sigma}\hat{g} \\ \phi \to \phi - \frac{\sigma}{\alpha}\end{array}\right\} \qquad e^{\alpha\phi}\hat{g} \to e^{\alpha\phi-\sigma}\cdot e^{\sigma}\hat{g}\,, \tag{4.25}$$

Implementing it in the full path integral leads to:

$$\begin{aligned}\mathcal{D}_{e^{\sigma}\hat{g}}\left(\phi - \frac{\sigma}{\alpha}\right)\mathcal{D}_{e^{\sigma}\hat{g}}\,(b)\,\mathcal{D}_{e^{\sigma}\hat{g}}\,(c)\,e^{-\mathcal{S}\left(\phi-\frac{\sigma}{\alpha},e^{\sigma}\hat{g}\right)} &= \\ &= \mathcal{D}_{\hat{g}}\,(\phi)\,\mathcal{D}_{\hat{g}}\,(b)\,\mathcal{D}_{\hat{g}}\,(c)\,\mathcal{D}_{\hat{g}}\,(\chi)\,e^{-\mathcal{S}(\phi,\hat{g})} \\ &= \mathcal{D}_{e^{\sigma}\hat{g}}\,(\phi')\,\mathcal{D}_{e^{\sigma}\hat{g}}\,(b)\,\mathcal{D}_{e^{\sigma}\hat{g}}\,(c)\,\mathcal{D}_{e^{\sigma}\hat{g}}\,(\chi)\,e^{-\mathcal{S}(\phi',\hat{g})}\,.\end{aligned} \tag{4.26}$$

The total conformal anomaly must vanish by consistency

$$0 \;=\; c_{tot} \;=\; c(\phi) + d - 26 \;=\; 1 + 3\tilde{Q}^2 + d - 26 \tag{4.27}$$

This leads to

$$\tilde{Q} \;=\; \sqrt{\frac{25-d}{3}}\,. \tag{4.28}$$

On the other hand, the vertex operator $e^{\alpha\phi}$ must be a $(1,1)$ conformal field in order for it to be integrated invariantly. This fixes the conformal weight

$$\Delta\left(e^{\alpha\phi}\right) \;=\; -\frac{1}{2}\alpha\left(\alpha - \tilde{Q}\right) \;=\; 1\,, \tag{4.29}$$

which in turn determines $\tilde{Q}$ in terms of α;

$$\tilde{Q} \;=\; \alpha \,+\, \frac{2}{\alpha} \tag{4.30}$$

Unfortunately, this shows that Liouville carries a central charge of *at least* 25, so that the only matter which can be naively coupled to it has $c < 1$, which is not enough for a string interpretation.

4.2 Canonical quantization and first levels of the spectrum

The two-dimensional locally supersymmetric action generalizing the one used above for the bosonic string reads (once the auxiliary fields have been eliminated)

$$\begin{aligned}S \;=\; -\frac{1}{8\pi}\int d^2\xi\sqrt{h}\Big[&h^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\eta_{\mu\nu} + 2i\bar{\psi}^{\mu}\gamma^a\partial_a\psi^{\nu}\eta_{\mu\nu} \\ &- i\bar{\chi}_a\gamma^b\gamma^a\psi^{\mu}\left(\partial_b X_{\mu} - \frac{i}{4}\bar{\chi}_b\psi_{\mu}\right)\Big].\end{aligned} \tag{4.31}$$

This action includes a scalar supermultiplet $(X^{\mu}, \psi^{\mu}, F^{\mu})$, where F^{μ} are auxiliary fields, and the two-dimensional gravity supermultiplet (e^a, χ_a, A), where again A is an auxiliary field.

The gravitino χ_a is a world-sheet vector-spinor. Using all the gauge symmetries of the action (reparametrizations, local supersymmetry and Weyl transformations) it is formally possible to reach the *superconformal gauge* where $h_{ab} = \delta_{ab}$ and $\chi_a = 0$. Off the critical dimension, however, there are obstructions similar, although technically more involved, to those present already in the bosonic string.

In this gauge, and using again complex notation, the action reads

$$\mathcal{S} = \frac{1}{2\pi} \int d^2z \Big\{ \partial_z X^\mu \partial_{\bar{z}} X_\mu + i \left(\psi^\mu_z \partial_{\bar{z}} \psi^{\mu z} + \psi^\mu_{\bar{z}} \partial_z \psi_{\mu \bar{z}} \right) \Big\} . \tag{4.32}$$

The energy-momentum tensor reads

$$T(z) \equiv T_{zz} = \frac{1}{2} \partial_z X^\mu \partial_z X_\mu + \frac{i}{2} \psi^\mu_z \partial_z \psi_{z\mu} , \tag{4.33}$$

and is holomorphic due to its conservation, $\partial_{\bar{z}} T_{zz} = 0$.

The supercurrent (associated to supersymmetry) reads

$$T^F_z = \frac{1}{2} \psi(z)^\mu \partial_z X_\mu . \tag{4.34}$$

This is again a holomorphic quantity, $\partial_{\bar{z}} T^F_z = 0$.

We saw earlier that ∂X^μ were conformal fields of weight $h = 1$, and as such admit a Fourier expansion

$$\partial X^\mu(z) = \sum \alpha^\mu_m z^{-m-1} . \tag{4.35}$$

The anti-holomorphic part enjoys a similar expansion

$$\partial X^\mu(\bar{z}) = \sum \bar{\alpha}^\mu_m \bar{z}^{-m-1} . \tag{4.36}$$

Similarly, the fermionic coordinates, being conformal fields with $h = \frac{1}{2}$, can be expanded as

$$\psi^\mu(z) = \sum b^\mu_n z^{-n-1/2} . \tag{4.37}$$

For open strings we fix arbitrarily at one end

$$\psi_+(0,\tau) = \psi_-(0,\tau) , \tag{4.38}$$

and the equations of motion then allow for two possibilities at the other end

$$\psi_+(\pi,\tau) = \pm \psi_-(\pi,\tau) . \tag{4.39}$$

The two sectors are called Ramond (for the + sign) and Neveu-Schwarz (for the − sign).

In the closed string case, fermionic fields need only be periodic up to a sign.

$$\psi_\mu(e^{2\pi i} z) = \pm \psi_\mu(z) . \tag{4.40}$$

Antiperiodic fields are said to obey the R(amond) boundary conditions; periodic ones are said to obey N(eveu)-S(chwarz) ones. Please note that, owing to the half-integer conformal weight of these fields, periodic fields in the plane correspond to antiperiodic fields in the cylinder.

This leads, in the closed string sector, to four possible combinations (for left as well as right movers), namely: (R,R), (NS,NS), (NS,R), (R,NS).

The Fourier components of the energy-momentum tensor (that is the generators of the Virasoro algebra) can be computed to be

$$L_m = \frac{1}{2\pi i}\oint dz\, T(z) z^{m+1} = \frac{1}{2}\left[\sum_n :\alpha^\mu_{-n}\alpha^\mu_{m+n}: + \sum_{r\in\mathbb{Z},\mathbb{Z}+\frac{1}{2}}\left(r+\frac{m}{2}\right) :b^\mu_{-r}b^\mu_{m+r}:\right], \tag{4.41}$$

and the modes of the supercurrent can be similarly shown to be equal to

$$G_r = \frac{1}{2\pi i}\oint dz\, T_F z^{r+1/2} = \sum_n \alpha^\mu_{-n} b^\mu_{r+n}\,. \tag{4.42}$$

The realitiy conditions then imply as usual

$$L_n^\dagger = L_{-n}\,, \qquad G_r^\dagger = G_{-r}\,. \tag{4.43}$$

In terms of the generators of the Virasoro algebra, the Hamiltonian (that is, the generator of dilatations on the plane, or, translations in τ on the cylinder) reads

$$H = L_0 + \bar{L}_0\,. \tag{4.44}$$

The unitary operator $U_\delta \equiv e^{i\delta(L_0-\bar{L}_0)}$ implements spatial translations in σ in the cylinder

$$U_\delta^\dagger X^\mu(\tau,\sigma)U_\delta = X^\mu(\tau,\sigma+\delta)\,. \tag{4.45}$$

This transformation should be immaterial for closed strings, which means that in that case we have the further constraint

$$L_0 = \bar{L}_0\,. \tag{4.46}$$

Covariant, *old fashioned*, canonical quantization can then be shown to lead to the canonical commutators

$$\begin{aligned}
[x^\mu, p^\nu] &= i\eta^{\mu\nu}\,,\\
[\alpha^\mu_m, \alpha^\nu_n] &= m\delta_{m+n}\eta^{\mu\nu}\,,\\
[b^\mu_r, b^\nu_s] &= \eta^{\mu\nu}\delta_{r+s}\,,
\end{aligned} \tag{4.47}$$

(all other commutators vanishing) and similar relations for the commutator of the $\bar{\alpha}$'s.

This means that we can divide all modes in two sets, positive and negative, and identify one of them (for example, the positive subset) as annihilation operators for harmonic oscillators. On the cylinder, the modding for the NS fermions is half-integer and integer for the R fermions. We can now set up a convenient Fock vacuum (in a sector with a given center of mass momentum), p^μ, by

$$\begin{cases} \alpha^\mu_m \mid 0, p^\mu\rangle = 0\,, & (m>0)\,, \\ b^\mu_r \mid 0, p^\mu\rangle = 0\,, & (r>0)\,, \\ P^\mu \mid 0, p^\mu\rangle = p^\mu \mid 0, p^\mu\rangle . & \end{cases} \tag{4.48}$$

There are a few things to be noted here: the first one is that $\alpha^0_{-m} \mid 0\rangle$ $(m>0)$ are negative-norm, ghostly, states, i.e. $\langle 0 \mid \alpha^0_m \alpha^0_{-m} \mid 0\rangle = -m\langle 0 \mid 0\rangle < 0$. The second thing to note is that, in the case of the R sector, the zero mode operators span a Clifford algebra, $\{b^\mu_0, b^\nu_0\} = \eta^{\mu\nu}$, so that they can be represented in terms of Dirac γ-matrices.

Recalling again that the Virasoro algebra reads

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m\left(m^2-1\right)\delta_{m+n}\,, \tag{4.49}$$

where the central charge is equal to the dimension of the external spacetime, $c = d$, it is plain that we cannot impose the vanishing of the L_n's as a strong constraint. Instead we can impose them as a weak constraint. Therefore we impose

$$NS: \begin{cases} L_m \mid Phys\rangle = 0\,, & m>0 \\ (L_0 - a) \mid Phys\rangle = 0\,, & \\ G_r \mid Phys\rangle = 0\,, & r \geq 1/2 \\ \left(L_0 - \bar{L}_0\right) \mid Phys\rangle = 0\,, & \end{cases} \tag{4.50}$$

$$R: \begin{cases} L_m \mid Phys\rangle = 0\,, & m \geq 0\,, \\ G_r \mid Phys\rangle = 0\,, & r \geq 0\,, \end{cases}$$

Spurious states are by definition states of the form

$$L_{-n} \mid \chi\rangle + \bar{L}_{-n} \mid \bar{\chi}\rangle\,, \tag{4.51}$$

for $n > 0$ since they are orthogonal to all physical states. Now, all physical states which are also spurious are called null. This then means that the observable Hilbert space is equivalent to the physical states modulo null states (Because the latter decouple from any amplitude). Let us now work out, for illustrative purposes, the first levels of the bosonic open string spectrum. We shall repeat this exercise from different points of view because each one illuminates a particular aspect of the problem.

For open strings we impose Neumann conditions at the boundary of the string world-sheet (meaning physically that no momentum is leaking out of the string), i.e.

$$X'_\mu = 0\,, \qquad (\sigma = 0, \pi)\,. \tag{4.52}$$

The appropriate solution then reads

$$X^\mu(\sigma,\tau) \,=\, x^\mu \,+\, \frac{1}{\pi T} p^\mu \tau \,+\, \frac{i}{\sqrt{\pi T}} \sum_{n\neq 0} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)\,. \tag{4.53}$$

A calculation of the Hamiltonian then shows that in this case

$$H \,=\, L_0 \,=\, \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{-n}^\mu \alpha_n^\mu\,. \tag{4.54}$$

When representing the open string in the upper-half plane, the tangent projection of the energy-momentum tensor, $T_{ab}t^b$, is still conserved. The condition that no energy-momentum should flow out of the string is then that, on the boundary $(y = 0)$

$$T_{ab} t^a n^b \,=\, 0\,, \tag{4.55}$$

or, using $t = \frac{\partial}{\partial x}$ and $n = \frac{\partial}{\partial y}$,

$$T_{zz} \,=\, \overline{T}_{\bar z \bar z}\,, \qquad (Im\, z \,=\, 0)\,. \tag{4.56}$$

The Fock ground state for the open string $|\, 0\, k\rangle$ satisfies

$$0 \,=\, (L_0 \,-\, a)\,|\, 0\, k\rangle \,=\, \left(2k^2 \,-\, a\right)\,|\, 0\, k\rangle\,, \tag{4.57}$$

which implies that $M^2 = -k^2 = -a/2 = -a/\alpha'$.

The next level is given by the states $|\, e, k\rangle = e_\mu \alpha_{-1}^\mu \,|\, 0\, k\rangle$. Imposing that it is a physical state

$$0 = (L_0 \,-\, a)\,|\, e\, k\rangle \,=\, (2k^2 + \alpha_{-1}\cdot\alpha_1 - a)\,|\, e\, k\rangle \,=\, (2k^2 + 1 - a)\,|\, e\, k\rangle\,, \tag{4.58}$$

$$0 = L_1\,|\, e\, k\rangle \,=\, 2k\cdot\alpha_1\,|\, e\, k\rangle \,=\, 2k\cdot e\,|\, e\, k\rangle\,, \tag{4.59}$$

so that the state has to satisfy $M^2 = \frac{1-a}{\alpha'}$ and $k \cdot e = 0$. The only available spurious state is obtained when $e \sim k$

$$L_{-1}\,|\, 0, k\rangle \,=\, 2k\cdot\alpha_{-1}\,|\, 0, k\rangle\,, \tag{4.60}$$

which is also null if $k^2 = 0$, which happens only if $a = 1$. There are several possibilities:

- $a < 1$, $M^2 > 0$: there are no null states and the constraint $k \cdot e = 0$ removes the negative norm timelike polarization. This corresponds to a massive vector.

- $a = 1$, $M^2 = 0$: this means that $k^\mu = (\omega, \vec{0}, \omega)$. The physical states are $e^\mu \sim k^\mu$, (the null states), and $D - 2$ states of the form $\vec{e}_T$.

- $a > 1$, $M^2 < 0$: this seems to be unacceptable.

We shall see momentarily that only for $a = 1$ and $D = 26$, the *old coveriant quantization* coincides with BRST- and lightcone quantization.

4.3 Physical (non-covariant) light-cone gauge and GSO projection

It is actually possible to solve all constraints (so that the remaining variables are all physical) by going to the light-cone gauge in which the x^+ target-*time* is related to the world-sheet time variable τ by

$$X^+ \;=\; \alpha' p^+ \tau\,. \tag{4.61}$$

The $T_{ab} = 0$ constraint is explicitly solved by

$$\partial_\pm X^- \;=\; \frac{1}{\alpha' p^+}\left(\left(\partial_\pm X^i\right)^2 + i\psi^i_\pm \partial_\pm \psi^i_\pm\right)\,. \tag{4.62}$$

This then means that both X^+ and X^- are actually eliminated in the Light-Cone gauge (X^+ by definition, and X^- as a consequence of the above).

The *mass*^{2} operator reads (for closed strings)

$$M^2 \;=\; 2P^+P^- - P_T^2 \;=\; \frac{2}{\alpha'}\left(\sum_{n>0}\left(\alpha^i_{-n}\alpha^i_n + \bar\alpha^i_{-n}\bar\alpha^i_n\right) + \sum_r r\left(b^i_{-r}b^i_r + \bar b^i_{-r}\bar b^i_r\right) - 2a\right)\,, \tag{4.63}$$

and the Hamiltonian reads

$$H \;=\; P^+P^- \;=\; P_T^2 \;+\; \frac{M^2}{2} \;=\; L_0 \;+\; \bar L_0 - 2a\,. \tag{4.64}$$

4.3.1 Open string spectrum and GSO projection

It is now neccessary to discriminate between the different sectors.

NS sector. The ground state, i.e. the oscillator vacuum, satisfies $\alpha' M^2 \mid 0, p^i\rangle = -a \mid 0, p^i\rangle$. The first excited state $b^i_{-1/2} \mid 0, p^i\rangle$ is a $(d-2)$ vector and Lorentz invariance then tells us that $M^2 = 0 = 1/2 - a_{NS}$, fixing the value for a_{NS} to be $a_{NS} = 1/2$. As a consequence we see that the mass of the vacuum state is given by:

$$\alpha' M^2_{vac} \;=\; -\frac{1}{2}\,. \tag{4.65}$$

Remembering that a_{NS} was a normal ordering constant we can calculate

$$a_{NS} \;=\; -\frac{d-2}{2}\left\{\sum_{n=0}^{\infty} n \;-\; \sum_{r=1/2}^{\infty} r\right\} \;=\; \frac{d-2}{16}\,, \tag{4.66}$$

resulting in the well-known $d = 10$.[9]

[9] In order to evaluate the normal ordering constant we used ζ- regularization, i.e. the vacuum energy is given by $E_\pm \equiv \pm\frac{d_T}{2}S(\alpha)$, where the upper sign stands for bosons, and the lower one for

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R sector. Let $\mid a\rangle$ be a state such that $b_0^\mu \mid a\rangle = \frac{1}{\sqrt{2}}(\gamma)^{\mu a}{}_b \mid b\rangle$, meaning that it defines an SO(1, 9) spinor with a priori $2^5 = 32$ complex components, which after imposing the Majorana-Weyl condition are reduced to 16 real components (8 on shell). This number is exactly the number that can be created with the oscillators b_0^i. The root of this fact is the famous triality symmetry of SO(8) between the vector and the two spinor representations, the three of having dimension 8.

There are then two possible chiralities: $\mid a\rangle$ or $\mid \bar{a}\rangle$, and $\alpha' M^2 = 0$, because oscillators do not contribute, and $a_R = 0$.

We are free to attribute arbitrarily a given fermion number to the vacuum.

$$(-)^F \mid 0\rangle_{NS} = - \mid 0\rangle_{NS} . \tag{4.70}$$

This gives $(-)^F = -1$ for states created out of the NS vacuum by an even number of fermion operators. Gliozzi, Sherk and Olive (GSO) [66] proposed to truncate the theory, by eliminating all states with $(-)^F = -1$. It is highly nontrivial to show that this leads to a consistent theory, but actually it does, moreover, it is spacetime supersymmetric. We demand then that all states obey $(-)^F_{NS} = 1$, thus eliminating the tachyon. This is called the GSO projection. On the Ramond sector, we define a generalized chirality operator, such that it counts ordinary fermion numbers and on the R vacuum,

$$(-)^F \mid a\rangle = \mid a\rangle , \qquad (-)^F \mid \bar{a}\rangle = - \mid \bar{a}\rangle , \tag{4.71}$$

There is now some freedom: to be specific, on the R sector we can demand either $(-)^F_R = 1$ or $(-)^F_R = -1$.

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fermions, and

$$S(\alpha) \equiv \sum_{n=0}^{\infty}(n+\alpha) = \zeta(-1,\alpha) = -\frac{1}{2}(\alpha^2 - \alpha + 1/6) , \tag{4.67}$$

Hardy in his famous book [65] on divergent series starts from properties one would like for any series to hold: define $\sum_{n=0}^{\infty} a_n = S(a)$, then what we want is

1. $\sum k a_n = k S(a)$,
2. $\sum(a_n + b_n) = S(a) + S(b)$ and
3. that if we split the sum we should have $\sum_{n=1}^{\infty} a_n = S(a) - a_0$.

In this case one can see that 1. and 3. are satisfied, but that the second is not, since upon splitting one finds that

$$\sum_{n=0}^{\infty}(n+a) = \sum_{n=0}^{\infty} n + \alpha \sum_{n=0}^{\infty} 1 = -\frac{1}{12} + \alpha\frac{1}{2} , \tag{4.68}$$

so that one misses out on the qudratic part. The sum is however uniquely defined by

$$S(0) = -\tfrac{1}{12} = \zeta(-1) , \qquad S(\alpha - 1) = S(\alpha) + \alpha - 1 . \tag{4.69}$$

There is a rationale for all this: the tachyon vertex operator in two-dimensional superspace is

$$V(p) = \int dz d\theta : e^{ipX(z,\theta)} : \tag{4.72}$$

which is *odd* with respect to $\psi \to -\psi$. Instead, the vector vertex operator is given by:

$$V_\mu = \int dz d\theta : i\mathcal{D}X_\mu e^{ipX(z,\theta)} : \tag{4.73}$$

which is even. To say it in other words: if we accept as physical the vector boson state, GSO amounts to projecting away all states related to it through an *odd* number of fermionic ψ-oscillators.

4.3.2 Closed string spectrum

The difference with the above case is that one has to consider as independent sectors the left and right movers.

(NS,NS) sector: The composite ground state is the tensor product of the NS vacuum for the right movers and the NS vacuum for the left-movers, and as such it drops out after the GSO projection. The first states surviving the GSO projection, that is $(-1)^F = (1,1)$, are

$$\bar{b}^i_{-1/2} \mid 0\rangle_L \otimes b^j_{-1/2} \mid 0\rangle_R\,. \tag{4.74}$$

Decomposing this in irreducible representations of the little group SO(8) yields $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ showing that it is equivalent to a scalar ϕ, the singlet, an antisymmetric 2-form field $B_{\mu\nu}$, the $\mathbf{28}$, and a symmetric 2-tensor field $g_{\mu\nu}$, the $\mathbf{35}$.

(R,R) sector, type IIA. The massless states are of the form $(-1)^F = (-1,1)$

$$\mid \bar{a}\rangle_L \otimes \mid b\rangle_R\,, \tag{4.75}$$

and decompose as $\mathbf{8}_v \oplus \mathbf{56}_v$, corresponding to a vector field, a one-form A_1, and a 3-form field, A_3.

(R,R) sector, type IIB. The massless states, with $(-1)^F = (1,1)$ are

$$\mid a\rangle_L \otimes \mid b\rangle_R\,, \tag{4.76}$$

and they decompose as $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_s$ corresponding to a pseudo scalar, χ, a 2-form field, A_2, and a selfdual 4-form field, A_4.

(R,NS) sector, Type IIA. The first GSO surviving states, with $(-1)^F = (-1,1)$, are

$$\mid \bar{a}\rangle_L \otimes b^i_{-1/2} \mid 0\rangle_R\,, \tag{4.77}$$

and they decompose as $\mathbf{8}_s \oplus \mathbf{56}_s$.

(R,NS) sector, Type IIB. The first GSO surviving states, with $(-1)^F = (1,1)$ are

$$| \, a\rangle_L \otimes b^i_{-1/2} \mid 0\rangle_R \,, \tag{4.78}$$

and they decompose as $\mathbf{8}_c \oplus \mathbf{56}_c$.

(NS,R) sector, Type IIA. The first GSO surviving states, with $(-1)^F = (1,-1)$ are

$$\bar{b}^i_{-1/2} \mid 0\rangle_L \otimes \mid \bar{a}\rangle_R \,, \tag{4.79}$$

and decompose as $\mathbf{8}_s \oplus \mathbf{56}_s$.

(NS,R) sector, Type IIB. The first GSO surviving states, with $(-1)^F = (1,1)$, are

$$\bar{b}^i_{-1/2} \mid 0\rangle_L \otimes \mid \, a\rangle_R \,, \tag{4.80}$$

and decompose as $\mathbf{8}_c \oplus \mathbf{56}_c$. The $\mathbf{56}_c$ corresponds to two gravitinos.

4.4 BRST quantization and vertex operators

Let us first consider the bosonic string. We know that the total conformal anomaly is given by

$$c_{total} \;=\; c(X) \;+\; c(ghosts) \;=\; d - 26 \,. \tag{4.81}$$

We define a classically consererved fermion number, the ghost number, operator (and the corresponding definition of the ghost number of a field) through

$$\begin{cases} j_{gh}(z) = - : bc :, \\ j_{gh}(z)\phi(w) = \dfrac{N_g}{z-w}\phi(w)\,. \end{cases} \tag{4.82}$$

The BRST charge is then defined as usual in gauge theories (see for example [67])

$$\mathcal{Q} \;=\; \oint \frac{dz}{2\pi i}\, j_{BRST}(z) \;=\; \oint \frac{dz}{2\pi i}\, c(z)\left[T(z) \;+\; \frac{1}{2}T_{gh}(z)\right]. \tag{4.83}$$

It is possible to show that $\mathcal{Q}^2 = 0$ iff $d = 26$.

From the point of view of the gauge-fixed covariant theory, physical states correspond to the BRST cohomology (that is: BRST closed states modulo BRST exactness).

Now, it is not difficult to show that those correspond to bosonic primary fields of conformal dimension (1,1):

$$\oint \frac{dz}{2\pi i} j_{BRST}(z)V(w) \;=\; \oint \frac{dz}{2\pi i} c(z)\left[\frac{hV(w)}{(w-z)^2} \;+\; \frac{\partial V(w)}{z-w}\right] = h\partial c V \;+\; c\partial V \,, \tag{4.84}$$

which is kosher iff $h = 1$, because in that case it is equal to $\partial(cV)$, which vanishes upon integration over the insertion point of the vertex operator on the Riemann surface representing the world-sheet of the string.

The usual SL$(2,\mathbb{C})$ ghost vacuum is defined as for any conformal field by

$$\begin{cases} b_n \mid 0\rangle_{gh} = 0\,, & n \geq -1\,, \qquad (b = \sum b_n z^{-n-2})\,, \\ c_n \mid 0\rangle_{gh} = 0\,, & n \geq 2\,, \qquad (c = \sum c_n z^{-n+1})\,. \end{cases} \tag{4.85}$$

We know that canonical quantization yields

$$b_0^2 \;=\; c_0^2 \;=0\,, \qquad \{b_0, c_0\} \;=\; 1\,. \tag{4.86}$$

On the other hand, for any conformal field

$$[L_n, \phi_m] = [n(h-1) - m]\phi_{n+m}\,, \tag{4.87}$$

so that in particular we can lower the L_0 value of the SL$(2,\mathbb{C})$ vacuum, $\mid 0\rangle_{gh}$, using

$$[L_0, c_1] \;=\; -c_1\,. \tag{4.88}$$

The eigenvalue of L_0 is preserved by c_0, i.e.

$$\begin{aligned} &[L_0, c_0] = 0\,, \\ L_0 c_1 \mid 0\rangle_{gh} \;&=\; -c_1 \mid 0\rangle_{gh} + c_1 L_0 \mid 0\rangle_{gh}\,, \end{aligned} \tag{4.89}$$

All this implies that the true lowest weight states are

$$\begin{aligned} c_1 \mid 0\rangle_{gh} \;&\equiv c(0) \mid 0\rangle_{gh} \;&= |{\downarrow}\rangle\,, \\ c_0 c_1 \mid 0\rangle_{gh} \;&= -c\partial c(0)\;\mid 0\rangle_{gh} &= |{\uparrow}\rangle\,. \end{aligned} \tag{4.90}$$

It is not difficult to check that these new states are both of zero norm, $\langle{\uparrow}|{\uparrow}\rangle = \langle{\downarrow}|{\downarrow}\rangle$ $= \,_{gh}\langle 0 \mid c_{-1}c_1 \mid 0\rangle_{gh} = \,_{gh}\langle 0 \mid c_{-1}c_1(b_0c_0 + c_0b_0) \mid 0\rangle_{gh} = 0$; but we can write

$$\langle{\downarrow}|{\uparrow}\rangle \;=\; \langle 0 \mid c_{-1}c_0c_1 \mid 0\rangle \;\equiv\; 1\,. \tag{4.91}$$

A small calculation then shows that

$$b_0\;|{\uparrow}\rangle = b_0c_0c_1 \mid 0\rangle \;=\; c_1 \mid 0\rangle \;= |{\downarrow}\rangle\,, \tag{4.92}$$

$$c_0\;|{\downarrow}\rangle = c_0c_1 \mid 0\rangle \;= |{\uparrow}\rangle\,, \tag{4.93}$$

from which one can infer that

$$N_g\left(|{\uparrow}\rangle\right) = \frac{1}{2}\,, \qquad N_g\left(|{\downarrow}\rangle\right) = -\frac{1}{2}\,, \qquad N_g\left(\mid 0\rangle\right) = -\frac{3}{2}\,, \tag{4.94}$$

that is: the vacuum carries three units of ghost number. It is quite easy to prove that, denoting by $z_{ij} \equiv z_i - z_j$,

$$\langle 0 \mid c(z_1)c(z_2)c(z_3) \mid 0\rangle \;=\; z_{23}z_{12}z_{13}\,. \tag{4.95}$$

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The appropriate projector is[10]

$$\mathcal{P} = \mid 0\rangle\langle 0 \mid c_{-1}c_0c_1\,, \tag{4.96}$$

which obviously obeys

$$\mathcal{P}^2 = \mathcal{P}\,. \tag{4.97}$$

Physical states correspond then to vertex operators of the type

$$V(z) \;=\; V_X(z)\cdot c(z)\,, \tag{4.98}$$

where $h\left(V_X(z)\right) = 1$.[11]

It is instructive to rederive some facts of the mass spectrum, using BRST techniques, at least for the bosonic string, to avoid technicalities: on physical states we need to have $b_0 \mid \psi\rangle = 0$, which can be used to derive[12]

$$\{\mathcal{Q}, b_0\} \mid \psi\rangle \;=\; \left(L_0^X + L_0^{gh}\right) \mid \psi\rangle \;=\; \left(2k^2 + L - 1\right) \mid \psi\rangle \;=\; 0\,, \tag{4.99}$$

implying that $M^2 = \frac{L-1}{2}$.

At oscillator level zero we can write:

$$0 \;=\; \mathcal{Q}\,|\!\downarrow 0\,k\rangle \;=\; (2k^2-1)c_0\,|\!\downarrow 0\,k\rangle\,, \tag{4.100}$$

implying $k^2 = \frac{1}{2}$ so that this state is the tachyon in the open string sector.

At oscillator level $L = 1$, there are 26+2 possible states having $M^2 = 0$ and they can be parametrized by

$$\mid \psi\rangle \;=\; (e\cdot a_{-1} \,+\, \beta b_{-1} \,+\, \gamma c_{-1})\,|\!\downarrow 0\,k\rangle\,. \tag{4.101}$$

Imposing that they be physical states, i.e. $\mathcal{Q} \mid \psi\rangle = 0$, leads to the constraints

$$k^2 \;=\; 0\,, \qquad k\cdot e \;=\; \beta \;=\; 0\,, \tag{4.102}$$

thus reducing the number of independent components to 26. In order to find the true number of independent states, we need to throw out the exact states

$$\mathcal{Q} \mid \chi\rangle \;=\; 2\,(k\cdot e' c_{-1} \,+\, \beta' k\cdot\alpha_{-1})\,|\!\downarrow 0\,k\rangle \tag{4.103}$$

This means that $c_{-1}\,|\!\downarrow 0\,k\rangle$ is exact and that $e_\mu \sim e_\mu + 2\beta' k_\mu$, yielding 24 positive norm states for a massless vector.

BRST reduces to old covariant for ghosts in the ground state, i.e. every cohomology class includes a state of this form.

[10] Please note that $\mathcal{P}' \equiv\mid 0\rangle\langle 0 \mid$ is null, since $\langle 0 \mid 0\rangle = \langle 0 \mid c_0b_0 + b_0c_0 \mid 0\rangle = 0$.

[11] For example: in case of the tachyon this implies $k^2 = 2$.

[12] We shall denote by L the *level* of a given state, i.e. the number of creation operators needed for the creation of the state out of the vacuum.

4.4.1 Superstrings

The $N = 1$ superconformal algebra is usually represented by the quantity $\hat{c}(= \frac{2}{3}c)$, and is generated by definition by the stress tensor T and the supercurrent T_F, with the OPEs

$$T(z)T(w) = \frac{\frac{3}{4}\hat{c}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \tag{4.104}$$

$$T(z)T_F(w) = \frac{\frac{3}{2}T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w} + \cdots \tag{4.105}$$

$$T_F(z)T_F(w) = \frac{\frac{1}{4}\hat{c}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots \tag{4.106}$$

In the superstring there are two basic superconformal fields

$$T(z) = -\frac{1}{2} : \partial X^\mu \partial X_\mu : -\frac{1}{2} : \partial\psi^\mu \cdot \psi_\mu : , \tag{4.107}$$

$$T_F = -\frac{1}{2} : \psi_\mu \partial x^\mu : , \tag{4.108}$$

and the ghost action corresponds to the superfields $B = \beta + \theta b$ and $C = c + \theta\gamma$. This then means that the contribution of the combined ghost system to the conformal anomaly is $c_{gh} = -26 + 11 = -15$:

$$T_{gh} = -2 : b\partial c : - : \partial b \cdot c : -\frac{3}{2} : \beta\partial\gamma : -\frac{1}{2} : \partial\beta \cdot \gamma : , \tag{4.109}$$

$$T_{F,gh} = \frac{1}{2} : b\gamma : - : \partial\beta \cdot c : -\frac{3}{2} : \beta\partial c : . \tag{4.110}$$

For any superalgebra we can expand

$$T_F(z) = \frac{1}{2} \sum_{r \in \mathbb{Z}+NSR} z^{-\frac{3}{2}-r} G_r , \tag{4.111}$$

where NSR is $0(\frac{1}{2})$ for R (NS, resp.). Any Ramond field is periodic on the cylinder, but on the plane

$$V^R\left(e^{2\pi i} z\right) = -V^R(z) . \tag{4.112}$$

NS Sector. There exists now a finite subalgebra OSp$(1 \mid 2)$, generated by $[L_0, L_{\pm 1}, G_{\pm 1}]$, with $G^2_{-\frac{1}{2}} = L_{-1}$ and its vacuum is defined by

$$\begin{aligned} L_n \mid 0\rangle &= 0 : n \geq -1 \ , \qquad & \langle 0 \mid L_n &= 0 : n \leq 1 , \\ G_r \mid 0\rangle &= 0 : r \geq -\frac{1}{2} \ , \qquad & \langle 0 \mid G_r &= 0 : r \leq \frac{1}{2} , \end{aligned} \tag{4.113}$$

R Sector. The superconformal anomaly implies that on the plane $G_0^2 = L_0 - \frac{\hat{c}}{16}$ (whereas on the cylinder $G_0^2 = L_0$ and besides, $[G_0, L_0] = 0$, which means that there are now two different ground states, $\mid h^+\rangle$ which is degenerate with $\mid h^-\rangle = G_0 \mid h^+\rangle$:

$$G_0 \mid h^-\rangle \;=\; 0\,, \qquad G_0^2 \mid h^\pm\rangle \;=\; 0\,. \tag{4.114}$$

They both obey

$$L_0 \mid h^\pm\rangle = \frac{\hat{c}}{16} \mid h^\pm\rangle, \tag{4.115}$$

such that

$$\langle h^- \mid h^-\rangle \;=\; \langle h^+ \mid G_0^2 \mid h^+\rangle \;=\; 0\,. \tag{4.116}$$

We then infer by completeness the existence of spin fields

$$\mid h^\pm\rangle \;=\; S^\pm(0) \mid 0\rangle\,, \tag{4.117}$$

which furthermore satisfy

$$\hat{G}_0 S^+(z) \;=\; S^-(z)\,. \tag{4.118}$$

The OPEs then, by consistency, neccessarily read

$$T_F(w)S^+(z) = \frac{1}{2}\frac{1}{(w-z)^{3/2}}S^-(z)\,, \tag{4.119}$$

$$T_F(w)S^-(z) = \frac{1}{2}\left(h - \frac{\hat{c}}{16}\right)\frac{1}{(w-z)^{3/2}}S^+(z)\,, \tag{4.120}$$

These fields interpolate between the NS and the R sectors, because they transform the NS groundstate into the R groundstate. In a somewhat symbolic notation

$$\phi_f^{NS}\left(e^{2\pi i}z\right)S^\pm(0) \;=\; -\phi_f^{NS}(z)\,S^\pm(0)\,. \tag{4.121}$$

Let us finally mention that the fact that the vacuum carries three units of ghost charge is related to the fact that in order to bosonize the (b,c) system, we had to introduce a background charge $Q(b,c) = -i\frac{3}{2}$. Had we done the same exercise for the superconformal (β,γ) system, we would have seen that at tree level, the total superconformal ghost charge adds to -2, $Q(\beta,\gamma) = +1$, but it can be traded between different vertex operators within a BRST invariant correlation function, symbollically

$$\langle 0 \mid e^{3\sigma(0)-2\phi(0)} \mid 0\rangle \;=\; 1\,, \tag{4.122}$$

where σ bosonizes the (b,c) and ϕ bosonizes the (β,γ) system. This is the basic reason why it is neccessary to have a different representative of each vertex operator in every ghost number sector, and to combine them in any correlator so that they match as above. This procedure was called *picture changing* by Friedan, Martinec and Shenker. See [16, 54] for further details.

4.5 Scattering amplitudes and the partition function

We are now prepared to compute some amplitudes (for more details see [4, 14]). The simplest thing would be the open string tachyon-tachyon scattering amplitude. The tree level (lowest order) contribution will be given by the correlator

$$A_4 \equiv \left\langle \int dz_3 c(z_1) V_1 c(z_2) V_2 V_3 c(z_4) V_4 \right\rangle , \tag{4.123}$$

where the vertex operator for the tachyon is given by

$$V_i \equiv : e^{ik^{(i)}_\mu X^\mu(z_i)} : . \tag{4.124}$$

The ghost factors are neccessary in order to cancel the ghost charge of the vacuum. We can arbitrarily choose their positions: it can be seen that this is equivalent to correctly taking into account the three conformal Killing vectors of the sphere

$$CKV : V = \left(\alpha + \beta z + \gamma z^2\right) \partial . \tag{4.125}$$

This leads to the basic string amplitude

$$A_4 = \int dz_3\, z_{12} z_{14} z_{24} \prod_{i<j} e^{p_i \cdot p_j \log z_{ij}} , \tag{4.126}$$

where $z_{ij} = z_i - z_j$. Since this should not depend on the positions at which we have placed the ghosts, we can choose

$$\left.\begin{array}{r} z_1 \to \infty \\ z_2 \to 1 \\ z_4 \to 0 \end{array}\right\} z_{12}^{p_1 \cdot p_2 + 1} z_{14}^{p_1 \cdot p_4 + 1} z_{24}^{p_2 \cdot p_4 + 1} z_{13}^{p_1 \cdot p_3} z_{23}^{p_2 \cdot p_3} z_{34}^{p_3 \cdot p_4} = (1 - z_3)^{p_2 \cdot p_3} z_3^{p_3 \cdot p_4} , \tag{4.127}$$

where we have used the fact that $\sum_i p_i = 0$ and $p_i \cdot p_i = 2$. As a result one ends up with the Veneziano amplitude:

$$A_4^{(g=0)} = \int_0^1 dz_3 \ (1 - z_3)^{p_2 \cdot p_3} z_3^{p_3 \cdot p_4} . \tag{4.128}$$

If we recall the usual definition of the Mandelstam parameters, $s \equiv -(p_1 + p_2)^2$ $t \equiv -(p_2 + p_3)^2$ and remember that we are dealing with tachyons, so that $p_i^2 = 2$, we can notice that the Veneziano amplitude can be written as Euler's Beta function, or, in terms of Gamma functions, as

$$A_4^{(g=0)} = \frac{\Gamma(-1 - t/2)\Gamma(-1 - s/2)}{\Gamma(-2 - (t + s)/2)} . \tag{4.129}$$

Now, it is well known that Euler's $\Gamma(z)$ has poles for all negative integers, $z \in \mathbb{Z}^-$. Here this translates into poles in the Veneziano amplitude at integer values of the *Regge trajectory*

$$\alpha(s) = 1 + s/2 \tag{4.130}$$

or, in the t-channel,

$$\alpha(t) = 1 + t/2 \tag{4.131}$$

The fact that the amplitude (4.129) does not change by interchanging t and s signals the (much sought for) property of *duality* (in the old sense of the word) for physical amplitudes. There is a superb historical introduction on this, and related, matters in the first chapter of [4].

The one-loop (genus one in the closed string case, that is a torus) amplitude is called the partition function.

Any two-dimensional torus (that is $\mathbb{R}^2/\Lambda$, where Λ is a two-dimensional lattice) can be put, by a conformal transformation, in the canonical form represented in the figure, where the lattice is generated by the complex numbers $1 = (1, 0)$ and $\tau = (\tau_1, \tau_2)$. Let us use the coordinates $z = \xi_1 + i\xi_2$ and $ds^2 = |dz|^2$. The periodicity conditions on the bosonic fields are

$$\begin{cases} \phi(\xi_1 + 1, \xi_2) \;= \phi(\xi_1, \xi_2)\,, \\ \phi(\xi_1, \xi_2 + \tau_2) = \phi(\xi_1 + \tau_1, \xi_2)\,. \end{cases} \tag{4.132}$$

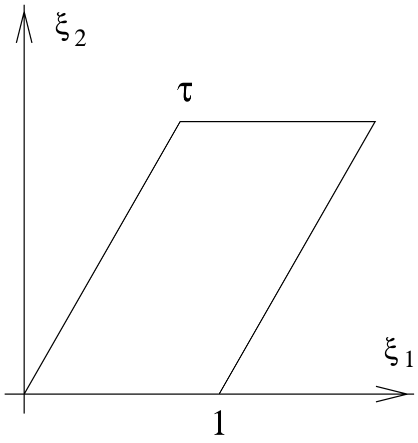

Figure 4: A general torus

Instead of twisted boundary conditions we could as well use normal periodic ones, but with a different metric, namely

$$z = \xi_1 + \tau\xi_2\,, \qquad ds^2 = |dz|^2\,, \qquad g = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \tag{4.133}$$

The total area is given by

$$A = \int \sqrt{g}\, d^2\xi = \tau_2\,. \tag{4.134}$$

The Beltrami differentials, corresponding to constant deformations (zero modes) which can not be represented as derivatives of periodic functions, are given by $\delta z = \delta\tau\xi_2 = \delta\tau\, \frac{z-\bar{z}}{2i\tau_2}$. The non-gauge deformations of the metric are given by

$$\delta g_{zz} = -\frac{\delta\tau}{2i\tau_2} = \frac{i}{2\tau_2}\delta\tau \;\equiv\; \mu_{zz}\delta\tau\,. \tag{4.135}$$

The quadratic differentials, on the other hand, are trivial constants: $\phi = (dz)^2$, $(\phi, \phi) = \tau_2$ and $(\mu, \phi) = 1$,

$$\frac{|\langle\phi \mid \mu\rangle|^2}{\langle\phi \mid \phi\rangle} = \frac{1}{\tau_2}\,. \tag{4.136}$$

The CKV are given by the condition $\partial\bar{z} = 0$, and the no-poles condition then means that $\bar{z} = c$, where c is a constant. This means that

$$Vol\,(CKV) = \tau_2\,. \tag{4.137}$$

The partition function then reads (using $det'\Delta = \tau_2|\eta|^4$):[13]

$$\mathcal{Z} = \int_F \frac{d^2\tau}{(\tau_2)^2}\,(\tau_2)^{-12}\,|\eta(\tau)|^{-48} = \int_F \frac{d^2\tau}{(\tau_2)^2}\,\chi(\tau,\bar{\tau})\,, \tag{4.139}$$

where use has been made of the Dedekind function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\ , \qquad q \equiv e^{2\pi i\tau}\,. \tag{4.140}$$

There are, however, diffeomorphisms not connected to the identity which we still have to factor out. They are generated by Dehn twists: once we identify two sides of the square to build a cylinder, we twist one of the boundaries by an integer multiple of 2π before identification. It is not difficult to show that they are generated by

$$S: \qquad \tau \to \tau' \equiv -1/\tau\,, \tag{4.141}$$

and

$$T: \qquad \tau \to \tau' \equiv \tau + 1\,. \tag{4.142}$$

That is, modular parameters related by any combination of the preceding transformations, (S,T) are diffeomorphic, and, as such, they should not be counted twice. This set of transformations actually forms a group, called the *modular group*, $\Gamma \equiv SL(2,\mathbb{Z})$. All this physically means that we have to restrict the integration in the one-loop amplitude to a fundamental region, F, of the modular group, which is by definition such that any $\tau \in H$ (the upper complex plane) can be mapped into F by a unique element of the modular group and, besides, no two different elements in F are gauge equivalent. A convenient choice of F is the intersection of

$$-\frac{1}{2} \le \tau_1 \le +\frac{1}{2}\,, \tag{4.143}$$

with

$$|\tau| > 1\,. \tag{4.144}$$

Naturally, in order for this restriction to F to be consistent, the *integrand* has to be modular invariant (which is, let us recall, a remainder of two-dimensional general covariance). This property is one of the most important symmetries of string theory, lying at the root of its consistency, in different incarnations: anomaly cancellation,

[13] The easiest way to get this result is to perform a path integral calculation in the light cone gauge using the normal mode expansion

$$X^i = \sum_{n_1,n_2\in\mathbb{Z}} X^i_{n_1,n_2} e^{2\pi i n_1\xi_1} e^{2\pi i n_2\xi_2} \tag{4.138}$$

and computing the ensuing determinant using ζ-function techniques (which in this particular case lead to a simple Epstein function).

spacetime supersymmetry, finiteness of the amplitudes and so on. This basic fact is quite easy to establish, using the fact that the *Poincaré* measure $\frac{d^2\tau}{(\tau_2)^2}$ is modular invariant by itself. Let us assume that the amplitude is given by some integral over the upper half plane, H,

$$A = \int_H \frac{d^2\tau}{(\tau_2)^2} M(\tau)\,, \tag{4.145}$$

If M is modular invariant (and only then), we can write

$$\begin{aligned} A &= \sum_{\gamma\in\Gamma} \int_{\gamma F} \frac{d^2\tau}{(\tau_2)^2} M(\tau) \\ &= \sum_{\gamma\in\Gamma} \int_F \frac{d^2\tau}{(\tau_2)^2} M(\tau) = \mathrm{vol}(\Gamma) \int_F \frac{d^2\tau}{(\tau_2)^2} M(\tau)\,. \end{aligned} \tag{4.146}$$

Please note that due to modular invariance the dangerous ultraviolet region $\tau_2 \to 0$ is excluded completely from the onset: any possible divergence can always be interpreted as an infrared one.

Under this $\mathrm{SL}(2,\mathbb{Z})$ transformation one finds that

$$\begin{aligned} d^2\tau &\to |c\tau + d|^{-4} d^2\tau\,, & \eta(\tau+1) &= e^{i\pi/12}\eta(\tau)\,, \\ \tau_2 &\to |c\tau+d|^{-2}\tau_2, & \eta(-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\eta(\tau)\,, \end{aligned} \tag{4.147}$$

showing that the partition function indeed enjoys modular invariance.

There is a very important Hamiltonian interpretation of the preceding corresponding to a propagation during a time τ_2 and performing an spatial twist of τ_1. Remembering that $L_0 + \bar{L}_0$ generates time translations, and that $L_0 - \bar{L}_0$ generates spatial translations, we are led to

$$\chi \sim Tr\, e^{2\pi i \tau_1 (H_R - H_L)} e^{-2\pi\tau_2(H_R+H_L)} = \int \frac{d^{24}p}{(2\pi)^{24}}\, e^{-2\pi p^2 \tau_1} e^{4\pi\tau_2} Tr\, \bar{q}^{H_L} q^{H_R}\,. \tag{4.148}$$

In order to evaluate this, we calculate

$$Tr\, \bar{q}^{N_L} q^{N_R} = \frac{1}{(\tau_2)^{12}} e^{4\pi\tau_2} \prod_{n=1}^{\infty} (1-\bar{q}^n)^{-24} (1-q^n)^{-24} = \frac{1}{(\tau_2)^{12}} \mid \eta(\tau) \mid^{-48}\,. \tag{4.149}$$

The meaning of this becomes clear when we remember that

$$H_L = \frac{p_T^2}{2} + N_L - 1\,, \qquad H_R = \frac{p_T^2}{2} + N_R - 1\,, \tag{4.150}$$

after which we can expand in a Laurent series

$$\chi \sim \frac{1}{|q|^2} + \frac{24}{q} + \frac{24}{\bar{q}} + 576 + \cdots \tag{4.151}$$

The quadratic pole represents the tachyon and the 576 ($= 299 + 276 + 1$) are the massless string states. The whole string spectrum can be easily reconstructed in this way.

4.5.1 Spin structures

Seiberg and Witten [68] first realized that on a torus each fermion can be characterized by the signs it gets when coming back to the same point after encircling one of the two homology cycles

$$\begin{aligned}\psi(\xi_1+1,\xi_2) &= \pm\psi(\xi_1,\xi_2)\,,\\ \psi(\xi_1,\xi_2+1) &= \pm\psi(\xi_1,\xi_2)\,.\end{aligned} \tag{4.152}$$

The corresponding contributions to the partition functions are given by:

$$\begin{aligned} A^{++}(\tau) &= \eta_{++} Tr\, e^{2\pi i\tau H_R}(-)^F\,, \qquad & A^{+-}(\tau) &= \eta_{+-} Tr\, e^{2\pi i\tau H_R}\,,\\ A^{--}(\tau) &= \eta_{--} Tr\, e^{2\pi i\tau H_{NS}}\,, \qquad & A^{-+}(\tau) &= \eta_{-+} Tr\, e^{2\pi i\tau H_{NS}}(-)^F\,,\end{aligned} \tag{4.153}$$

where the Hamiltonians are given by (remembering that the normal ordering constants can be computed by the general formula as $\frac{1}{3} = -\frac{d_T}{2}\zeta(-1,0)$ and $-\frac{1}{6} = -\frac{d_T}{2}\zeta(-1,1/2)$)

$$H_R = \sum_{m=1}^{\infty} m b^i_{-m} b^i_m + \frac{1}{3}\,, \tag{4.154}$$

$$H_{NS} = \sum_{r=\frac{1}{2}}^{\infty} r b^i_{-r} b^i_r - \frac{1}{6}\,. \tag{4.155}$$

Elliptic Theta functions are defined for arbitrary characteristics as [69]

$$\Theta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(0\mid\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi(n+\theta)^2\tau + 2\pi i(n+\theta)\phi}\,. \tag{4.156}$$

The Jacobi elliptic functions are particular cases of the above

$$\begin{aligned} \Theta\begin{bmatrix}1/2\\ 1/2\end{bmatrix} &= \theta_1\,, \qquad & \Theta\begin{bmatrix}1/2\\ 0\end{bmatrix} &= \theta_2\,,\\ \Theta\begin{bmatrix}0\\ 0\end{bmatrix} &= \theta_3\,, \qquad & \Theta\begin{bmatrix}0\\ 1/2\end{bmatrix} &= \theta_4\,,\end{aligned} \tag{4.157}$$

Using which we can rewrite the A's as

$$\begin{aligned} A^{++}(\tau) &= \eta_{++}\frac{\theta_1^4(0\mid\tau)}{\eta^4(\tau)} = 0\,, \qquad & A^{+-}(\tau) &= \eta_{+-}\frac{\theta_2^4(0\mid\tau)}{\eta^4(\tau)}\,,\\ A^{--}(\tau) &= \eta_{--}\frac{\theta_3^4(0\mid\tau)}{\eta^4(\tau)}\,, \qquad & A^{-+}(\tau) &= \eta_{-+}\frac{\theta_4^4(0\mid\tau)}{\eta^4(\tau)}\,.\end{aligned} \tag{4.158}$$

A^{++} is the odd spin structure and the others are the even spin structures (where the label even or odd, stands for the number of zero modes of the Dirac operator $(+2\mathbb{Z})$).

Under the one-loop modular group $Sl(2,\mathbb{Z})$ theta functions transform amongst themselves, so that fixing, for example, the phase $\eta_{--} = 1$, all other phases are uniquely determined by modular invariance [70]

$$\begin{aligned} A(\tau) &= \frac{1}{2\eta^4}\left\{\theta_3^4 - \theta_4^4 - \theta_2^4 + \eta_{++}\theta_1^4\right\} \\ &= Tr\, e^{2\pi i\tau H_{NS}}\left(\frac{1}{2}\left(1-(-1)^F\right)\right) - Tr\, e^{2\pi i\tau H_R}\left(\frac{1}{2}\left(1-\eta_{++}(-1)^F\right)\right) \end{aligned} \tag{4.159}$$

where one should note that the minus sign in the definition of the trace over the NS d.o.f. implies that the NS vacuum has a negative value for the fermion number operator. Jacobi's *Equatio identica satis abstrusa* ensures that this is identically zero, and in this sense is equivalent to supersymmetry: this allows for a reinterpretation of the GSO projection as one-loop modular invariance.

It is curious that if instead of asking for separate L and R modular invariance, we had summed over the same boundary conditions L and R, we would not have gotten a spacetime supersymmetric action, but the bosonic Dixon-Harvey model instead [70].

There is a nice argument by Polyakov [3] on the relationship between modular invariance and spacetime anomalies: the Ward identity associated to conformal transformations reads

$$\langle \delta S \phi(z_1)\cdots\phi(z_n)\rangle = \sum_i \langle \phi(z_1)\cdots\delta\phi(z_i)\cdots\phi(z_n)\rangle\,, \tag{4.160}$$

where

$$\delta S = \int T\bar{\partial}\epsilon\,, \tag{4.161}$$

and

$$\delta\phi = \epsilon\partial\phi + h\partial\epsilon\phi\,. \tag{4.162}$$

The basic origin of the OPE from this point of view stems from choosing ϵ in such a way that $\bar{\partial}\epsilon = \delta^{(2)}(z)$; that is, $\epsilon = \frac{1}{z}$. On a torus, the best we can do is to choose $\epsilon = \zeta(z)$, where the ζ function of Weierstraß is defined by $\zeta'(z) = -\mathcal{P}(z)$, and begins its Laurent expansion with a simple pole at the origin. It is not doubly periodic however, but rather

$$\zeta(z + 2\omega + 2\omega') = \zeta(z) + 2\eta + 2\eta'\,, \tag{4.163}$$

where ω and ω' are the two half-periods, and

$$\eta\omega' - \eta'\omega = \frac{\pi i}{2}\,. \tag{4.164}$$

This fact induces a change in the modular parameter

$$\tau \to \tau + \frac{\pi i}{2\omega(\omega + \eta)}\,, \tag{4.165}$$

which causes a suplementary term in the Ward identity proportional to

$$\frac{\partial}{\partial\tau}\langle\phi_1\cdots\phi_n\rangle\,, \tag{4.166}$$

giving rise to boundary terms when integrated, in case the correlator is not modular invariant. All two-dimensional anomalies are related: a conformal anomaly can be disguised as a gravitational anomaly by a local counterterm.

4.6 Spectral flow and spacetime supersymmetry

The $N=2$ superconformal algebra is an extension of the preceding $N=1$ superconformal algebra, and is realized in the particular case when there is an $U(1)$ charge, generated by $J(z)$, with $h[J]=1$, such that

$$T_F^+(z)T_F^-(w) = \frac{c/12}{(z-w)^3}+\frac{\frac{1}{4}J(w)}{(z-w)^2}+\frac{\frac{1}{4}T(w)+\frac{1}{8}\partial J(w)}{z-w}\,, \tag{4.167}$$

$$J(z)J(w) = \frac{c/3}{(z-w)^2}\,, \tag{4.168}$$

$$J(z)T_F^{\pm}(w) = \pm\frac{T_F^{\pm}(w)}{z-w}\,. \tag{4.169}$$

The spectral flow of Schwimmer and Seiberg [71] is given by the following transformations:

$$T^{\eta}(z) = T(z)+\frac{\eta J(z)}{z}+\frac{c\eta^2}{6z^2}\,, \tag{4.170}$$

$$T_F^{\eta\pm}(z) = z^{\pm\eta}\,T_F^{\pm}(z)\,, \tag{4.171}$$

$$J^{\eta}(z) = J(z)+\frac{c\eta}{3z}\,, \tag{4.172}$$

where $\eta\in[0,1)$. It is a fact that T^{η} satisfies the Virasoro algebra with the same c.

This physically means that all fermionic boundary conditions yield isomophic algebras. This is only possible because the supercurrent is 'real'

$$T_F^{\pm}\left(e^{2\pi i}z\right) = -e^{\mp 2\pi i\eta}T_F^{\pm}(z)\,. \tag{4.173}$$

Spectral flow obviously corresponds to

$$\begin{cases} h\to h^{\eta} = h+\eta q+\frac{c}{6}\eta^2\,, \\ q\to q^{\eta} = q+\frac{c}{3}\eta\,. \end{cases} \tag{4.174}$$

This fact strongly suggests that in order to have spacetime supersymmetry (That is, in order to be able to implement succesfully a GSO projection and build spin operators) one needs $N=2$ supersymmetry on the world-sheet.

4.7 Superstring taxonomy

We have now at hand all the necessary tools to build superstrings. In Hamiltonian language (that is, at fixed target-time), what we need for ensuring conformal invariance is a unitary CFT with $(c, \bar{c}) = (25, 25)$. The simplest possibility is the *bosonic string*, with $d = 26$. If we insist on spacetime supersymmetry with $(1, 1)$ world-sheet superconformal theories we find that

$$c = \bar{c} = \frac{3}{2}d - 26 + 11\,, \tag{4.175}$$

where the 11 part comes from the commuting ghosts fixing local supersymmetry. Imposing the constraint of vanishing conformal anomaly then leads to $d_{crit} = 10$.

There are several related theories in d=10. The simplest ones enjoy $N = 2$ (that is, 32 supercharges) supersymmetry, and come in two versions: one possibility is the so-called type IIA superstring, where the two gravitinos have oposite chirality, so that there is no chirality preferred, and the theory is non-chiral. This is the theory whose low energy limit is the dimensional reduction of N=1 Supergravity in d=11 dimensions.

Another possibility is that both gravitinos have the same chirality. This is the IIB theory, a chiral one, whose low energy limit is IIB supergravity in d=10.

There is also the possibility of having open strings. Open boundary conditions break the supersymmetry to $N = 1$ only. These theories are in general anomalous, unless a non-dynamical degree of freedom (a Chan-Paton index) is added to the ends, such that it belongs to SO(32). This is the Type I Superstring, with gauge group SO(32).

Narain [72] showed that in general one can have the dimensions of the left and right momentum lattices different, p and q say, with conformal invariance putting the restrictions that the lattice $(k, \bar{k})$ is even, unimodular and self-dual, with a metric of signature (p, q). (This means essentially that $k \cdot k' - \bar{k} \cdot \bar{k}' \in 2\mathbb{Z}$). The number of parameters associated to a Narain lattice is the dimension of the coset space $\mathrm{SO}(p, q)/(\mathrm{SO}(p) \otimes \mathrm{SO}(q))$; that is, pq. Narain, Sarmadi and Witten [72] further showed that these lattices can be interpreted as the effect of constant backgrounds for the spacetime metric as well as for the two-index field.

We can also cancel left and right anomalies in an independent way. This will eventually lead to two further string theories also in d=10: $E_8 \times E_8$ Heterotic and SO(32) Heterotic. There are then altogether five seemingly consistent ten dimensional string theories. For the simplest $(1, 0)$ 'heterotic' theory we can do the same matching

$$(c, \bar{c}) = (-15, -26)_{ghosts} + (15, 10)_{coord.} + (0, 16)_{extra}\,, \tag{4.176}$$

where the extra part can be shown to be an $E_8 \otimes E_8$ or SO(32) level 1 current algebra.

The Heterotic string has a 26 dimensional bosonic left moving sector, $X^\mu_L(\tau + \sigma)$, and an $N = 1$ supersymmetric rightmoving sector, $\psi^\mu_R, X^\mu_R(\tau - \sigma)$. Take $X^I_L(\tau + \sigma)$,

$I = 1..16$, to be compactified, in the sense that

$$P_L \in \Gamma_{16} \,:\, P_L^I = P_i\, e_i{}^I \quad , \quad P_i \in \mathbb{Z}\,. \tag{4.177}$$

The Hamiltonian turns out to be [72]

$$H_L = \frac{1}{2}P_T^2 + N_L + \frac{1}{2}P_I^2 - 1\,, \tag{4.178}$$

$$H_R = \frac{1}{2}P_T^2 + N_R + H^{NSR}\,. \tag{4.179}$$

The bosonized Right sector then lives in a Lorentzian lattice ($D_{5,1}$), putting together world-sheet fermionic imbeddings and ghosts (ψ^μ, β, γ), with weights $w_R = \lambda_R, q$ whereas the bosonic lattice is represented by $(\Gamma_{16})_L$.

Vertex operators involve

$$V = e^{iW_L \cdot X(\bar{z})} e^{i\lambda_R \cdot \phi(z)} e^{q\phi(z)}\,. \tag{4.180}$$

Locality, or rather the absence of branch cuts,

$$V_1(z)V_2(w) = (\bar{z} - \bar{w})^{W_{L_1}\cdot W_{L_2}} (z-w)^{\lambda_{R_1}\cdot\lambda_{R_2} - q_1 q_2}\, V_{1+2}\,, \tag{4.181}$$

imposes that

$$-W_{L_1}\cdot W_{L_2} + \lambda_{R_1}\cdot\lambda_{R_2} - q_1 q_2 \in \mathbb{Z}\,, \tag{4.182}$$

$$\Gamma_{16;5,1} \;\equiv\; \left(\Gamma_{16}\right)_L \otimes \left(D_{5,1}\right)_R\,. \tag{4.183}$$

This can further be elaborated to imply that Γ_{16} has to be an odd, selfdual Lorentzian lattice.

It is worth emphasizing that, although we do not have time to explain it in detail here, the fact that those string theories are anomaly free (which is related to conformal invariance) can also be studied from the effective field theory point of view: to each string theory corresponds a consistent, anomaly free supergravity [4, 73].

The low energy, long wavelength limit of this theory is N=1 Supergravity coupled to Yang-Mills in d=10. The bosonic part of the effective field theory of the heterotic string is given by

$$S^{het}_{(d=10)} = \frac{1}{32\pi}\int d^{10}x\sqrt{-g}e^{-\phi}\left[R(g) - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}F^I_{\mu\nu}F^{I\mu\nu}\right], \tag{4.184}$$

where $I = 1,\ldots,16$ represent the Abelian fields in the Cartan subalgebra of either $E_8\times E_8$ or SO(32), which are the only ones which *generically* will remain massless upon compactification to four dimensions.

On the other hand, the ordinary spacetime dimensional reduction of the 11-dimensional supergravity action gives the 10-dimensional IIA supergravity action, which can be written as

$$S^{10}_{IIA} = \frac{1}{2\kappa^2_{10}} \int d^{10}x\sqrt{-G}e^{-\phi} \left\{ R(G) - (\partial_\mu \phi)^2 + \frac{1}{2\cdot 3!}(H_{\mu\nu\rho})^2 \right.$$
$$\left. - \frac{1}{2\cdot 2!}e^{\phi}(F_{\mu\nu})^2 - \frac{1}{2\cdot 4!}e^{\phi}(J^{(4)}_{\mu\nu\rho\sigma})^2 \right\} + \frac{1}{4\kappa^2_{10}} \int K^{(4)} \wedge K^{(4)} \wedge B^{(2)} , \quad (4.185)$$

where $F^{(2)} \equiv dA^{(1)}$, $H^{(3)} \equiv dB^{(2)}$, $K^{(4)} \equiv dA^{(3)}$ and $J^{(4)} \equiv K^{(4)} + A^{(1)} \wedge H^{(3)}$.

One of the peculiar properties of supergravity is that there is another ten dimensional theory, called IIB, which is chiral, in the sense that the two gravitinos share the same chirality. The covariant equations of motion involve a self-dual five-form, and there is no known covariant action from which they can be derived. In the understanding that the self-duality constraint has to be imposed after the variation, a possible action is

$$S_{IIB} = \frac{1}{2\kappa^2_{10}} \left[\int d(vol) \left\{ e^{-\phi} \left(R - (\partial_\mu\phi)^2 + \frac{1}{2\cdot 3!} H^2_{\mu\nu\rho} \right) + \frac{1}{2}(\partial_\mu l)^2 \right.\right.$$
$$\left.\left. + \frac{1}{12}(H^{'}_{\mu\nu\rho} - l H_{\mu\nu\rho})^2 + \frac{1}{60} F_5^{+2} \right\} - \frac{1}{48} \int A^+_4 \wedge H_3 \wedge H'_3 \right] . \quad (4.186)$$

From this action it is obvious, just by counting powers of the dilaton, that the RR fields are $l \equiv A_0$, $A_2 \equiv B'_2$ and A^+_4.

It was known since a long time that there is, up to field redefinitions, only one action for $N = 1$ Supergravity coupled to Super Yang Mills in $d = 10$, the one describing the low energy limit of Type I open strings:

$$S_I = \frac{1}{2\kappa^2_{10}} \int d(vol) \left\{ e^{-\phi} \left[R - \partial_\mu \phi)^2 \right] - \frac{1}{4} e^{\phi/2} F^{a2}_{\mu\nu} + \frac{1}{12} H'^2_{\mu\nu\rho} \right\} \quad (4.187)$$

Here the three different powers of the dilaton reflect the different geometrical origin: the first terms come from the spherical topology, the gauge term comes from Chan-Paton factors attached to the disc (with $\chi = 1$), and the term with no dilaton comes from a $B'_2(RR)$ when this theory is viewed (as we shall see) as an orientifold of the IIB.

4.8 Strings in background fields

Up to now the strings have been propagating in ten-dimensional Minkowski space. We physically expect, however, that some kind of string condensate should explain spacetime curvature and, furthermore, that spacetime should spontaneously compactify to 4 dimensions. Unfortunately, these highly interesting topics are very difficult to study with first quantization techniques.

A less ambitious problem is to determine, given a passive spacetime background, whether strings can consistently propagate in it. Afterwards we could dream of including back-reactions in some self-consistent approximation.

Let us then assume that strings are propagating in a non-trivial background of the massless (NS) fields.

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2 z \left\{ \sqrt{h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \epsilon^{ab} b_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \right.$$
$$\left. + \frac{1}{2}\alpha' \sqrt{h} R^{(2)}\, \Phi(X) \right\} . \tag{4.188}$$

If we want to consider this as a QFT_2, the background fields have to be considered as field-dependent coupling-constants (cf. [88, 89, 90, 91]).

By performing a covariant (using Riemann normal-coordinates) background field expansion, the β-functions can be computed to be

$$\beta_{\mu\nu}(g) = R_{\mu\nu} - \frac{1}{4} H^2_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi\,, \tag{4.189}$$
$$\beta_{\mu\nu}(b) = -\frac{1}{2} \nabla^\rho H_{\mu\nu\rho} + \frac{1}{2} \nabla^\rho \Phi \cdot H_{\mu\nu\rho}\,, \tag{4.190}$$
$$\beta(\Phi) = -R + \frac{1}{12} H^2 - 2\nabla^2\Phi + (\nabla\Phi)^2 + \frac{D-26}{3\alpha'} . \tag{4.191}$$

The conformal anomaly of the σ-model, $c \equiv \beta(\Phi)$, must be constant by consistency. That this is actually true, can be seen by noting that it is the integrated version of the other two and the Bianchi identity (see e.g. [89]).

It is a remarkable feat that the β-functions can be derived from the action

$$\mathcal{S} = \int d^D x \sqrt{g}\, e^{-\Phi} \left[R - \frac{1}{12} H^2 + (\nabla\Phi)^2 - 2\nabla^2\Phi + \frac{D-26}{3\alpha'} \right] . \tag{4.192}$$

This is perhaps the most important (together with the similar results we derived at the beginning for kappa symmetry) of all results linking spacetime physics with world-sheet properties. Classical solutions of the above spacetime action represent possible *string vacua*; ground states of quantum strings.

Hull and Townsend remarked that on-shell one can make field redefinitions $X^i \to X^i + f^i(X)$ such that all one needs to have for a vanishing $T^\alpha{}_\alpha$ up to field redefinition is

$$\beta_{\mu\nu}(g) = \nabla_\mu \tilde{V}_\nu + \nabla_\nu \tilde{V}_\mu\,, \tag{4.193}$$
$$\beta_{\mu\nu}(b) = \tilde{V}^\rho H_{\mu\nu\rho}\,, \tag{4.194}$$

for some field $\tilde{V}_\mu$. Vanishing of the conformal anomaly demands in addition that

$$\tilde{V}_\mu = \partial_\mu \Phi\,. \tag{4.195}$$

We can transform from the string frame to the Einstein frame, by means of the field redefinition

$$g_{\mu\nu} = e^{\frac{2}{D-2}\Phi}\, g^{(E)}_{\mu\nu}\,. \tag{4.196}$$

The action (4.192) in the Einstein frame reads

$$\mathcal{S}_D = \int d^Dx\sqrt{g^{(E)}}\left\{R(g^{(E)}) + \frac{1}{D-2}\left(\nabla^{(E)}\Phi\right)^2 - \frac{1}{12}e^{-\frac{4}{D-2}\Phi}H^2_{(E)} + e^{\frac{2}{D-2}\Phi}\frac{D-26}{3\alpha'}\right\}. \tag{4.197}$$

Classical solutions to this action give conformally invariant σ-models to $\mathcal{O}(\alpha')$.

If we write $\Phi = \langle\Phi\rangle + \hat{\Phi}$, any string amplitude containing $\exp(-\mathcal{S})$ scales as

$$e^{-\frac{\langle\Phi\rangle}{8\pi}\int\sqrt{h}R^{(2)}} = g_s^{2g-2}\,, \tag{4.198}$$

where $g_s = e^{\langle\Phi\rangle}$ and Euler's theorem tells us that

$$\frac{1}{8\pi}\int\sqrt{h}R^{(2)} = 2-2g\,. \tag{4.199}$$

This physically means that the vacuum expectation value (that is, the asymptotic value of the classical solution in most cases) of the dilaton field gives directly the string coupling constant, which is then promoted to a dynamical field.

Due to the presence of the background fields, the vertex operators are changed correspondingly: in general under a Weyl transformation $\gamma_{ab} \to e^{2\sigma}\gamma_{ab}$ there will be operator mixing

$$\frac{\delta}{\delta\sigma}\langle V_i\rangle = \sum_k \Delta_{ij}\langle V_j\rangle\,, \tag{4.200}$$

where Δ is the anomalous dimension matrix. The simplest example is the one corresponding to the tachyon determined by Callan and Gan [74]. They showed the anomalous dimension to be

$$\alpha'\left(\nabla^2 - 2\nabla_\mu\Phi\nabla^\mu\right)T(X)\,. \tag{4.201}$$

Physical vertex operators need to have conformal dimension (1,1), and are correspondingly solutions of the equations

$$\left(\nabla^2 - 2\nabla_\mu\Phi\nabla^\mu\right)T(X) = \frac{2}{\alpha'}T(X)\,, \tag{4.202}$$

recovering the old result that $T(X) = e^{ik\cdot X}$ in the simplest case when the background is flat space.

This can be put in a slightly different way by rescaling the metric as in eq. (4.196), namely

$$\left(\nabla^2 - \frac{2}{\alpha'}e^{\frac{4}{d-2}\Phi}\right)T(X) = 0\,. \tag{4.203}$$

Dilatons can then be thought of as locally rescaling the tachyon mass. It is also possible to argue that the quadratic fluctuation operator of the effective action can be regarded as the anomalous dimension operator for the massless state vertex operators.

5. T-duality, D-branes and Dirac-Born-Infeld

T-duality is the simplest of all dualities and the only one which can be shown to be true, at least in some contexts. At the same time it is a very stringy characteristic, and depends in an essential way on strings being extended objects. In a sense, the web of dualities rests on this foundation, so that it is important to understand clearly the basic physics involved. Let us consider strings living on an external space with one compact dimension, which we shall call y, with topology S^1.

5.1 Closed strings in S^1

1. Let us imagine that one of the dimensions of a given spacetime is a circle S^1 of radius R. The corresponding field in the imbedding of the string, which we shall call y (i.e. we are dividing the target-spacetime dimensions as (x^μ, y), where y parametrizes the circle), has then the possibility of winding around it:

$$y(\sigma + 2\pi, \tau) = y(\sigma, \tau) + 2\pi R m\,. \tag{5.1}$$

A closed string can close in general up to an isometry of the external spacetime.

The zero mode expansion of this coordinate (that is, forgetting about oscillators) would then be

$$y = y_c + 2p_c\tau + mR\sigma\,. \tag{5.2}$$

Canonical quantization leads to $[y_c, p_c] = i$, and single-valuedness of the plane wave $e^{iy_c p_c}$ enforces as usual $p_c \in \mathbb{Z}/R$, so that $p_c = \frac{n}{R}$.

The zero mode expansion can then be organized into left and right movers in the following way

$$\begin{aligned} y_L(\tau+\sigma) &= y_c/2 + \left(\frac{n}{R} + \frac{mR}{2}\right)(\tau+\sigma)\,, \\ y_R(\tau-\sigma) &= y_c/2 + \left(\frac{n}{R} - \frac{mR}{2}\right)(\tau-\sigma)\,. \end{aligned} \tag{5.3}$$

The mass shell conditions reduce to

$$\begin{aligned} m_L^2 &= \frac{1}{2}\left(\frac{n}{R} + \frac{mR}{2}\right)^2 + N_L - 1\,, \\ m_R^2 &= \frac{1}{2}\left(\frac{n}{R} - \frac{mR}{2}\right)^2 + N_R - 1\,. \end{aligned} \tag{5.4}$$

Level matching, $m_L = m_R$, implies that there is a relationship between momentum and winding numbers on the one hand, and the oscillator excess on the other

$$N_R - N_L = nm\,. \tag{5.5}$$

At this point it is already evident that the mass formula is invariant under

$$R \to R^* \equiv 2/R\,, \tag{5.6}$$

and exchanging momentum and winding numbers. This is the simplest instance of *T-Duality*.

2. The above transformation can be seen to lift to an automorphism to the CFT OPE namely

$$\begin{aligned} y(z) &\to y(z)\,, \\ \bar{y}(\bar{z}) &\to -\bar{y}(\bar{z})\,. \end{aligned} \tag{5.7}$$

The total momentum of the scalar field is defined as

$$\hat{p}. \equiv \frac{1}{4\pi}\oint(\partial\phi + \bar{\partial}\bar{\phi}). \tag{5.8}$$

so that vertex operators for momentum eigenstates are of the type

$$e^{ip(\phi+\bar{\phi})}\,, \tag{5.9}$$

because $\hat{p} : e^{ip(\phi+\bar{\phi})} := p : e^{ip(\phi+\bar{\phi})} :$. The total winding, on the other hand, is similarly written as

$$\hat{w}. \equiv \frac{1}{4\pi}\oint(\partial\phi - \bar{\partial}\bar{\phi}), \tag{5.10}$$

so that vertex operators for winding eigenstates are of the type

$$: e^{ik(\phi-\bar{\phi})} :, \tag{5.11}$$

enjoying $\hat{w} : e^{ik(\phi-\bar{\phi})} := k : e^{ik(\phi-\bar{\phi})} :$.

There is a simple argument showing that upon compactification, momentum eigenstates couple to the Kaluza-Klein gauge boson, whereas windings couple to gauge bosons coming from the reduction of the Kalb-Ramond field. Actually, considering the OPE

$$: \left(\partial X^\mu \bar{\partial} y \pm \partial y \bar{\partial} X^\mu\right) e^{ip\cdot X} : (z,\bar{z}) \; : e^{ik\cdot X(w,\bar{w})} e^{il(y(w)-\bar{y}(\bar{w}))} :\sim \; -k^\mu\, l\, (1 \mp 1)\,, \tag{5.12}$$

justifies the above claim.

3. Another point is that demanding the $(d-1)$-dimensional effective action to be invariant under this transformation we are forced to assume that

$$2\pi R^* e^{-2\phi^*} = 2\pi R e^{-2\phi}\,, \tag{5.13}$$

leading to the necessity of transforming the dilaton, already in this simple setting.

4. The integrand of the partition function of the bosonic string compactified on a circle (or a torus, for that matter) can be computed, for arbitrary genus, and shown to enjoy T-duality. We can expand the holomorphic differentials, ∂X, of the embedding in terms of the period matrix

$$\tau_{ij} = \int_{\beta_j} \omega_i \equiv (\tau_1)_{ij} + i\,(\tau_2)_{ij} \,, \tag{5.14}$$

where (α_i, β_j), $i, j = 1, \ldots, g$, is a canonical homology basis and the holomorphic differentials ω_i are normalized by

$$\int_{\alpha_i} \omega_j = \delta_{ij} \,. \tag{5.15}$$

We can then decompose the holomorphic differentials as

$$\partial y = \chi + \sum_n C_i \omega_i \,, \tag{5.16}$$

where χ is the exact part and the C's are determined by the number of times the string winds around the homology cycles, i.e.

$$\int_{a_i} dy = 2\pi n_i R \,, \qquad \int_{b_i} dy = 2\pi m_i R \,. \tag{5.17}$$

This allows for the computation of the integrand with the result [75]

$$\mathcal{F}_g(R) = \theta \begin{bmatrix} 0\,0 \\ 0\,0 \end{bmatrix} (0 \mid \Omega)\, \Lambda_g(\tau, \bar{\tau}) \,, \tag{5.18}$$

where Λ_g is the integrand of the decompactified partition function, and Riemann's theta function is defined by

$$\theta \begin{bmatrix} 0\,0 \\ 0\,0 \end{bmatrix} (0 \mid \Omega) \equiv \sum_{(n,m) \in \mathbb{Z}^{2g}} e^{i\pi\,(m\,n)\Omega(n\,m)^t} \,, \tag{5.19}$$

and

$$\Omega \equiv iR^2 \begin{pmatrix} \tau_1 \tau_2^{-1} \tau_1 + \tau_2 & -\tau_1 \tau_2^{-1} \\ -\tau_2^{-1} \tau_1 & \tau_2^{-1} \end{pmatrix} . \tag{5.20}$$

Now, under a symplectic transformation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, one finds [76]

$$\theta\left(0 \mid -\Omega^{-1}\right) = \det{}^{\frac{1}{2}} \left(\frac{\Omega}{i}\right) \theta\,(0 \mid \Omega) \,, \tag{5.21}$$

$$\mathcal{F}_g(R) = \left(\frac{1}{R^2}\right)^g \mathcal{F}_g\left(R^{-1}\right) . \tag{5.22}$$

For the whole sum, this implies ($\tilde{\kappa} = \kappa/R$)

$$\sum_g \kappa^{2g-2} \mathcal{F}_g(R) = R^{-2} \sum_g \tilde{\kappa}^{2g-2} \mathcal{F}_g(R^{-1}) \,. \tag{5.23}$$

5. Something special happens at the self-dual radius $R = R^* = \sqrt{2}$: there are four extra massless vectors proportional to

$$\partial x_\mu(z) e^{\pm i\sqrt{2}\bar{y}(\bar{z})} \quad , \quad \bar{\partial} x_\mu(\bar{z}) e^{\pm i\sqrt{2}y(z)} \,, \tag{5.24}$$

which, together with the two Kaluza-Klein vectors mentioned above, generate $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. Dine, Huet and Seiberg [77] were the first to realize that for generic values of the compactification radius this group gets reduced to the Abelian part $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$. They interpreted this as a stringy Higgs effect, and, as a consequence, T-duality must be included in the full stringy gauge symmetry. This was the first suggestion that T-duality ought to be exact, at least in perturbation theory, which was afterwards checked explicitly, at least in some examples [76].

6. In the supersymmetric case, owing to superconformal invariance, the CFT mapping is

$$\begin{aligned} x &\to x \,, \qquad \bar{x} \to -\bar{x} \,, \\ \psi &\to \psi \,, \qquad \bar{\psi} \to -\bar{\psi} \,. \end{aligned} \tag{5.25}$$

This means that chirality is reversed and that one goes from IIA at radius R to IIB at radius $2/R$.

7. The contribution to the partition function of momentum states goes as $\frac{1}{R}$; whereas the contribution of winding modes is linear in R. Let us consider a string moving in a p-dimensional torus, $T^p \equiv \mathbb{R}^p/(2\pi\Lambda)$, where Λ is a lattice. The zero mode contribution to the Polyakov integral is

$$\sum_{p,p' \in \Lambda} e^{-2\pi\left(\tau_2 p^2 + \frac{1}{\tau_2}(p' - \tau_1 p)^2\right)} \,. \tag{5.26}$$

We can now use the Poisson summation formula

$$\sum_{\mathbb{Z}} e^{-m^2 R^2} = \frac{\sqrt{2}}{R} \sum_{\mathbb{Z}} e^{-\frac{n^2\pi^2}{R^2}} \,, \tag{5.27}$$

to rewrite it as

$$\sum_{k,\bar{k},k-\bar{k} \in \Lambda^*} e^{i\pi(\tau k^2 - \bar{\tau}\bar{k}^2)} \,. \tag{5.28}$$

It can be shown that $(k, \bar{k})$ span a self-dual lattice with signature (p, p).

8. By compactifying a bosonic or superstring in $\Lambda(d, d)$ the full T-duality group is upgraded to $O(d, d; \mathbb{Z})$. Representing by E the sum of the background metric plus the background Kalb-Ramond, $E \equiv G + B$, the group acts in a projective way [78, 79]: given

$$g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{5.29}$$

then

$$E \to \frac{aE+b}{cE+d}\,. \tag{5.30}$$

This rather large discrete group is generated by the following transformations:

1. Discrete translations on the Kalb-Ramond field,

$$B_{\mu\nu} \to B_{\mu\nu} + \theta_{\mu\nu}\,. \tag{5.31}$$

   That is,

$$g \equiv \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, \tag{5.32}$$

   where $\theta\mu\nu \in \mathbb{Z}$.

2. A change of basis on the lattice, i.e.

$$E' \;\equiv\; AEA^t\,, \tag{5.33}$$

   that is

$$g \equiv \begin{pmatrix} A & 0 \\ 0 & A^{-1t} \end{pmatrix}. \tag{5.34}$$

   And, finally

3. Factorized duality (the analogous transformation to the duality we saw earlier on S^1):

$$g \equiv \begin{pmatrix} 1-e_i & e_i \\ e_i & 1-e_i \end{pmatrix}, \tag{5.35}$$

   where

$$(e_i)_{jk} \equiv \delta_{ji}\delta_{ki}\,. \tag{5.36}$$

In the case of the heterotic string, this gives $O(d+16, d; \mathbb{Z})$, because there is an extra Λ_{16} for the left movers.

5.2 T-duality for closed strings

There is a known way of proving T-duality for any string vacuum described by a sigma model such that the corresponding target-spacetime enjoys at least one isometry. The classic work was done by Buscher [92], but we shall follow the slightly different gauging approach first introduced by Roček and Verlinde [93]. Their formulation starts with the σ-model (with the Abelian isometry represented in adapted coordinates by $\theta \to \theta + \epsilon$)

$$S = \frac{1}{2\pi\alpha'}\int d^2\xi\left[\sqrt{h}\left(h^{ab}\partial_a x^\mu \partial_b x^\nu + R^{(2)}\phi(x)\right) + i\epsilon^{ab}b_{\mu\nu}\partial_a x^\mu \partial_b x^\nu\right]. \tag{5.37}$$

This means that in these adapted coordinates, (θ, x^i), the metric, torsion and dilaton fields are θ independent. The key point is to gauge the isometry by introducing some gauge fields A_a transforming as $\delta A_a = -\partial_a \epsilon$. Using a Lagrange multiplier term, the gauge field strength is required to vanish, enforcing the constraint that the gauge field is pure gauge. After gauge fixing the original model is then recovered.

Gauging the isometry and adding the Lagrange multiplier enforcing the condition that the gauge field strength vanishes leads to

$$\begin{aligned} S_{d+1} = \frac{1}{4\pi\alpha'} \int d^2\xi \Big[& \sqrt{h} h^{ab} \Big(g_{00}(\partial_a\theta + A_a)(\partial_b\theta + A_b) + 2g_{0i}(\partial_a\theta + A_a)\partial_b + x^i \\ & + g_{ij}\partial_a x^i \partial_b x^j \Big) + i\epsilon^{ab}(2b_{0i}\Big(\partial_a\theta + A_a)\partial_b x^i + b_{ij}\partial_a x^i \partial_b x^j \Big) + \\ & + 2i\epsilon^{ab}\tilde{\theta}\partial_a A_b + \alpha'\sqrt{h}R^{(2)}\phi(x) \Big] . \end{aligned} \tag{5.38}$$

The dual theory is obtained integrating out the A fields

$$A_a = -\frac{1}{g_{00}} \left(g_{0i}\partial_a x^i + i\frac{\epsilon_a{}^b}{\sqrt{h}} \left(b_{0i}\partial_b x^i + \partial_b\tilde{\theta} \right) \right), \tag{5.39}$$

and fixing $\theta = 0$,

$$\begin{aligned} \tilde{S} = \frac{1}{4\pi\alpha'} \int d^2\xi \Big[& \sqrt{h} h^{ab} \Big(\tilde{g}_{00}\partial_a\tilde{\theta}\partial_b\tilde{\theta} + 2\tilde{g}_{0i}\partial_a\tilde{\theta}\partial_b x^i + \tilde{g}_{ij}\partial_a x^i \partial_b x^j \Big) \\ & + i\epsilon^{ab}\Big(2\tilde{b}_{0i}\partial_a\tilde{\theta}\partial_b x^\alpha + \tilde{b}_{ij}\partial_a x^i \partial_b x^j \Big) + \alpha' \sqrt{h}R^{(2)}\phi(x) \Big] , \end{aligned} \tag{5.40}$$

where

$$\begin{aligned} \tilde{g}_{00} &= \frac{1}{g_{00}} , \\ \tilde{g}_{0i} &= \frac{b_{0i}}{g_{00}} , \qquad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}} , \\ \tilde{g}_{ij} &= g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}} , \\ \tilde{b}_{ij} &= b_{ij} - \frac{g_{0i}b_{0j} - g_{0j}b_{0i}}{g_{00}} . \end{aligned} \tag{5.41}$$

It so happens that Buscher's transformation can not be the whole story in presence of a nontrivial dilaton. Indeed, the dual model is not even conformally invariant in general, unless an appropriate transformation of the dilaton is included, namely

$$\tilde{\phi} = \phi - \frac{1}{2}\log k^2, \tag{5.42}$$

where $k = \frac{\partial}{\partial\theta}$ is the Killing vector field; in adapted coordinates, $k^2 = g_{00}$. In the pathintegral approach the way to obtain the correct dilaton shift yielding a conformally invariant dual theory can be seen as follows. In complex coordinates and on spherical world-sheets we can parametrize $A = \partial\alpha$, $\bar{A} = \bar{\partial}\beta$, for some 0-forms α, β on the manifold M. The change of variables from $A, \bar{A}$ to α, β produces a factor in the measure

$$\mathcal{D}A\mathcal{D}\bar{A} = \mathcal{D}\alpha\mathcal{D}\beta(\det\partial)(\det\bar{\partial}) = \mathcal{D}\alpha\mathcal{D}\beta(\det\Delta)\,. \tag{5.43}$$

Substituting $A, \bar{A}$ as functions of α, β and integrating over α, β, the following determinant emerges

$$(\det(\partial g_{00}\bar{\partial}))^{-1} \equiv det\Delta_{g_{00}}^{-1}\,. \tag{5.44}$$

In particular, the integration on β produces a delta-function

$$\delta\left(\bar{\partial}(g_{00}\partial\alpha + (g_{0i} - b_{0i})\partial x^i - \partial\tilde{\theta})\right)\,, \tag{5.45}$$

which when integrated over α yields the factor in the measure. What we finally get in the measure is then

$$\frac{\det\Delta}{\det\Delta_{g_{00}}}\,. \tag{5.46}$$

This formula provides a justification for Buscher's prescription for the computation of the determinant arising from the naive Gaussian integration. As we have just seen some care is needed in order to correctly define the measure of integration over the gauge fields. From the previous formula the dilaton shift is obtained in the following way. Writing g_{00} as $g_{00} = 1 + \sigma \approx e^\sigma$ we have:

$$\Delta_{g_{00}} = (1+\sigma)\Delta - h^{ab}\partial_a\sigma\partial_b\,. \tag{5.47}$$

Plugging this into the infinitesimal variation of Schwinger's formula

$$\delta\log\det\Delta = Tr\int_\epsilon^\infty dt\delta\Delta e^{-t\Delta}\,, \tag{5.48}$$

we obtain

$$\delta\log\det\Delta_{g_{00}} = -\int d^2\xi\sqrt{h}\Omega\left\langle\xi\left|e^{-\epsilon(\Delta+\sigma\Delta-h^{ab}\partial_a\sigma\partial_b)}\right|\xi\right\rangle\,, \tag{5.49}$$

where $\delta\Delta_{g_{00}} = -\Omega\Delta_{g_{00}}$ with $\delta h_{ab} = \Omega\delta_{ab}$. We can now use the standard heat kernel expansion [94]

$$\langle\xi|e^{-\epsilon D}|\xi\rangle = \frac{1}{4\pi\epsilon} + \frac{1}{4\pi}\left(\frac{1}{6}R^{(2)} - V\right)\,, \tag{5.50}$$

where

$$D \equiv \Delta - 2ih^{ab}A_a\partial_b + \left(-\frac{i}{\sqrt{h}}\partial_a(\sqrt{h}h^{ab}A_b\right) + h^{ab}A_aA_b) + V\,. \tag{5.51}$$

For $D = \Delta + \sigma\Delta - h^{ab}\partial_a\sigma\partial_b$ and after dropping the divergent term $1/4\pi\epsilon$ and the quadratic terms in σ, we obtain

$$\delta\log\det\Delta_{g_{00}} = -\frac{1}{8\pi}\int d^2\xi\sqrt{h}R^{(2)}\log g_{00}\,. \tag{5.52}$$

which inmediatly leads to

$$\det g_{00} = \exp\,(-\frac{1}{8\pi}\int d^2\xi\sqrt{h}R^{(2)}\log g_{00})\,, \tag{5.53}$$

implying $\tilde{\phi} = \phi - \frac{1}{2}\log g_{00}$. It is possible to interpret T-duality as a canonical transformation with generating functional [79]

$$\mathcal{F} = \frac{1}{2}\int_{D,\partial D=S^1} d\tilde{\theta}\wedge d\theta = \frac{1}{2}\int_{S^1}\left(\theta'\tilde{\theta} - \theta\tilde{\theta}'\right). \tag{5.54}$$

In the operator formalism [97], it is implemented by a Fourier transform of sorts, i.e.

$$\Psi_k\left[\tilde{\theta}(\sigma)\right] = \mathcal{N}(k)\int \mathcal{D}\theta(\sigma)\,e^{i\mathcal{F}[\theta,\tilde{\theta}]}\,\Phi_k\left[\theta(\sigma)\right]. \tag{5.55}$$

In the simplest free case, the Hamiltonians are

$$H = \frac{1}{2R}\left(\frac{\delta}{\delta\theta}\right)^2 + \frac{R}{2}\left(\theta'\right)^2\,, \tag{5.56}$$

$$\tilde{H} = \frac{R}{2}\left(\frac{\delta}{\delta\tilde{\theta}}\right)^2 + \frac{1}{2R}\left(\tilde{\theta}'\right)^2\,. \tag{5.57}$$

After a functional integration by parts, one obtains

$$\tilde{H}\Psi_k = \mathcal{N}(k)\int \mathcal{D}\theta\,e^{i\mathcal{F}}\left(\frac{1}{2R}\left(\frac{\delta}{\delta\theta}\right)^2 + \frac{R}{2}\left(\theta'\right)^2\right)\Phi_k\,. \tag{5.58}$$

5.3 T-Duality for open strings and D-branes

Let us begin with the simplest bosonic model, which, while allowing for the most interesting physical phenomena, is devoid of complications due to supersymmetry. We shall consider open and closed bosonic strings propagating in an arbitrary d-dimensional metric and (Abelian) gauge field. Wess-Zumino antisymmetric tensors are not consistent if the theory is non-orientable, but we shall include them nevertheless for the time being. In modern language (to be justified momentarily), we have a string interacting with a Dirichlet $(d-1)$-brane, and we consider non-trivial massless backgrounds in the longitudinal directions. In the neutral case (that is, the charge is opposite in both ends of the string), the action can be written as[14]

$$S = \frac{1}{4\pi}\int_\Sigma (g_{\mu\nu}\eta^{ab} + i\,b_{\mu\nu}\epsilon^{ab})\,\partial_a x^\mu\partial_b x^\nu + \frac{i}{2\pi}\int_{\partial\Sigma} n_a A_\mu\partial_b x^\mu\epsilon^{ab}\,. \tag{5.59}$$

The classification of allowed world-sheet topologies is much more complicated in the open case than in the more familiar closed one. The Euler characteristic can be written, in a somewhat symbolic form, as

$$\chi = 2 - 2g - b - c, \tag{5.60}$$

[14]We set $\alpha' = 1$ throughout.

where g is the number of handles, b the number of boundaries, and c the number of crosscaps. To the lowest order in string perturbation theory $\chi = 1$, only the disc D_2 and in the non-orientable case the crosscap or, to be more precise, the two-dimensional real projective plane $P_2(\mathbb{R})$, contribute. (To the following "one loop" order, corresponding to $\chi = 0$, we have the annulus A_2, the Möbius band M_2, and the Klein bottle K_2). In this section we shall only consider the leading contributions from the disc and the crosscap. The action will be invariant under a target isometry with Killing vector k^μ,

$$\delta_\epsilon x^\mu = \epsilon \, k^\mu(x) \,, \tag{5.61}$$

provided a vector ω_μ and a scalar φ exist, such that

$$\begin{aligned} \mathcal{L}_k \, g_{\mu\nu} &= 0 \,, \\ \mathcal{L}_k \, b_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \,, \\ \mathcal{L}_k \, A_\mu &= -\omega_\mu + \partial_\mu \phi \,, \end{aligned} \tag{5.62}$$

where $\mathcal{L}_k$ represents the Lie derivative with respect to the Killing vector. In the neutral case, it is clear that the boundary term representing the coupling of the background gauge field to the open string can be incorporated in the bulk action through the simple substitution[15]

$$b_{\mu\nu} \to B_{\mu\nu} = b_{\mu\nu} + F_{\mu\nu} \,. \tag{5.63}$$

Using the conditions on the background fields, it is easy to show that

$$\mathcal{L}_k \, B_{\mu\nu} = 0 \,. \tag{5.64}$$

In order to perform the duality transformation, it is convenient to rewrite the whole action as a redundant gauge system where the isometry is gauged. We must introduce a Lagrange multiplier to ensure that the auxiliary gauge field is flat. Minimal coupling is enough to construct the gauged action

$$\begin{aligned} S_{\text{gauged}} = {} & \frac{1}{4\pi} \int_\Sigma \left(g_{\mu\nu} \eta^{ab} + i \, B_{\mu\nu} \epsilon^{ab} \right) D_a x^\mu D_b x^\nu + \\ & + \frac{i}{4\pi} \int_\Sigma \tilde{x}^0 (\partial_a V_b - \partial_b V_a) \epsilon^{ab} - \frac{i}{2\pi} \int_{\partial\Sigma} \tilde{x}^0 V \,, \end{aligned} \tag{5.65}$$

where $D_a x^\mu = \partial_a x^\mu + k^\mu V_a$ and, in adapted coordinates, the Killing vector reads $k = \frac{\partial}{\partial x^0}$. The one-form associated to the gauge field is represented as $V = V_a dx^a$. The rôle of the boundary term is to convey invariance under translations of the Lagrange multiplier: $\tilde{x}^0 \to \tilde{x}^0 + C$. This was first derived by Dai, Leigh and Polchinski [98] and using this form in [96]. Boundary conditions are restricted by several

[15] When the charges at both ends of the string do not add to zero, one cannot get completely rid of the boundary term. We shall comment on a similar situation later in the main text.

physical requirements. The gauge parameter must have the same boundary conditions as the world-sheet fields (i.e. Neumann), in order for the isometry to be realized on the boundary of the Riemann surface. This has the obvious consequence that, if the gauge $V = 0$ were needed, it would be neccessary to impose on the gauge fields the boundary condition $n^a V_a \equiv V_n = 0$ (because this component can never be eliminated with gauge transformations obeying Neumann boundary conditions). It turns out, however, that in order to show the equivalence of (5.65) with the original model (5.59), the behaviour of $V|_{\partial\Sigma}$ is immaterial. The only way the previous action can now lead to the *unique* restriction $dV = 0$ on the gauge field is to restrict the variations of the Lagrange multiplier in such a way that $\delta\tilde{x}^0|_{\partial\Sigma} = 0$. In this way we are forced to impose Dirichlet boundary conditions $\tilde{x}^0 = C$ on the multiplier. Since the rest of the coordinates remain Neumann, a Dirichlet $(d-2)$-brane is obtained. Besides, this ensures gauge invariance. The last two terms can be combined into

$$\frac{i}{2\pi}\int_\Sigma -d\tilde{x}^0 \wedge V. \tag{5.66}$$

This means that the gauge field enters only algebraically in the action, and it can be replaced by its classical value (performing the Gaussian integration only modifies the dilaton terms)

$$V_a^{cl} = -\frac{1}{k^2}\Big(k^\mu g_{\mu\nu}\partial_a x^\nu + i\epsilon_a{}^b\partial_b\tilde{x}^0 + i\,\epsilon_a{}^b k^\mu B_{\mu\nu}\partial_b x^\nu\Big). \tag{5.67}$$

Particularizing now to adapted coordinates, $k = \frac{\partial}{\partial x^0}$ and choosing the dual gauge $x^0 = 0$, we get the dual model, whose functional form is exactly like the former, but with the backgrounds $\tilde{G}_{\mu\nu}$, $\tilde{B}_{\mu\nu}$ given in terms of the original ones through Buscher's formulas

$$\begin{aligned}
\tilde{G}_{00} &= \tilde{g}_{00} = \frac{1}{g_{00}}\,,\\
\tilde{G}_{0i} &= \frac{B_{0i}}{g_{00}}\,,\\
\tilde{G}_{ij} &= g_{ij} - \frac{g_{0i}g_{0j} - B_{0i}B_{0j}}{g_{00}}\,,\\
\tilde{B}_{0i} &= \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}}\,,\\
\tilde{B}_{ij} &= B_{ij} - \frac{g_{0i}B_{0j} - B_{0i}g_{0j}}{g_{00}}\,.
\end{aligned} \tag{5.68}$$

In deriving these expressions, some care must be exercised in choosing the appropriate variables in adapted coordinates. In this frame, the isometry is represented by simple translations $x^0 \to x^0 + \epsilon$. This means that the various backgrounds must be independent of x^0, up to target-space gauge transformations, which in this model are defined by

$$b \to b + d\lambda\,, \qquad A - \lambda\,, \tag{5.69}$$

where λ is an arbitrary one-form, in such a way that $B = b + dA$ is invariant. This gauge ambiguity is responsible for the occurence of the non-trivial Lie derivatives. In order to consistently reach the gauge $x^0 = 0$ in the closed string sector (world-sheets without boundaries), the torsion Lie derivative must be cancelled within the local patch of adapted coordinates. In fact, it is easily seen that the gauge transformation λ, defined as

$$\mathcal{L}_k \, \lambda = -\omega + d\varphi \,, \tag{5.70}$$

cancels both the ω and φ terms. In this gauge, all fields are locally independent of x^0. The behaviour of the dilaton under T-duality is always a subtle issue. In the present situation this is even more so, due to the fact that the metric is not a massless background of the open string, and one has to consider closed string corrections, thus driving the sigma model away from the conformally invariant point, in order to get a consistent Fischler-Susskind mechanism. There is, however, a neccesary condition for the equivalence of the two theories, and this is that the effective action must remain invariant. It turns out that this condition is sufficient to determine the dual dilaton to the value

$$\tilde{\phi} = \phi - \frac{1}{2} \log k^2 \,. \tag{5.71}$$

The invariance under translations of the dual model can sometimes be put to work in our benefit. Let us consider, for simplicity, the Wilson line $A_0 = \mathrm{diag}(\theta_1, \ldots, \theta_N)$, when only the coordinate x^0 is compactified in a circle of length $2\pi R$ in an otherwise flat background. The Wilson line itself, in the sector of winding number n is given by

$$\sum_{a=1}^{N} e^{2\pi i n R \theta_a} \,. \tag{5.72}$$

This term is reproduced in the dual model by simply taking into account the "total derivative term" coming from the Gaussian integration of the auxiliary gauge field, namely

$$\int_\Sigma dx^0 \wedge d\tilde{x}^0 \,, \tag{5.73}$$

which by Stokes' theorem can be written as $-\oint \tilde{x}^0 \, dx^0 = -C \, 2\pi n R$, where C is the constant value of the multiplier on the boundary. This value can depend on the (implicit) Chan-Paton indices of the world-sheet fields, and we recover with our techniques the result of Polchinski that the Wilson line considered induces in the dual model a series of D-branes with fixed positions determined by the θ_a parameters. Given any two different branes, there are open strings which can have one endpoint in each brane. The mass of those states is proportional to the distance between branes; this means that there is an enhancement of symmetry for coincident branes, which then should be described by some kind of non-Abelian generalization of DBI. An interesting observation is that the collective motion of the D-brane is already encoded in Buscher's formulas. To see this, note that the dual backgrounds differ from the standard duals without gauge fields by the terms

$$\begin{aligned}
\tilde{G}_{0i} &= \tilde{g}_{0i} - \tilde{g}_{00}\, \partial_i A_0 \,, \\
\tilde{G}_{ij} &= \tilde{g}_{ij} + \tilde{g}_{00}\, \partial_i A_0 \partial_j A_0 - \tilde{g}_{0i}\, \partial_j A_0 - \tilde{g}_{0j}\, \partial_i A_0 \,, \\
\tilde{B}_{ij} &= \tilde{b}_{ij} + F_{ij} + \tilde{b}_{0i}\, \partial_j A_0 - \tilde{b}_{0j}\, \partial_i A_0 \,.
\end{aligned} \tag{5.74}$$

Using these formulas it can be checked that the dual sigma model reduces to the standard one in terms of the backgrounds $\tilde{g}_{\mu\nu}$ and $\tilde{b}_{\mu\nu}$, provided we make the replacement

$$\tilde{x}^0 \to \tilde{x}^0 + A_0(x^i). \tag{5.75}$$

Thus, the gauge field component A_0 acquires the dual interpretation of the transverse position of the D-brane, as a function of x^i, which become longitudinal world-volume coordinates. The same result follows from a careful consideration of the boundary conditions. In the case of unoriented strings, the change from Neumann to Dirichlet conditions is supplemented by an orbifold projection in the spacetime which reverses the orientation of the world-sheet (the orientifold). This result follows easily from the T-duality mapping at the level of the conformal field theory. $\partial_z x(z) \to \partial_z x(z)$, $\partial_{\bar{z}} x(\bar{z}) \to -\partial_{\bar{z}} x(\bar{z})$. In order to study this question in a curved background, let us consider the lowest order unoriented topology, namely the crosscap $P_2(\mathbb{R})$. It is enough for our purposes to make in the boundary of the unit disc the identification of opposite points: $x^\mu(\sigma) = x^\mu(\sigma + \pi)$. (Here $\sigma \in (0, 2\pi)$ just parametrizes the boundary.) Dropping the zero mode, this yields the conditions (where the vectors n and t are the normalized outer normal and tangent vector to the world-sheet boundary and, correspondingly, $\partial_n \equiv n^a \partial_a$, and $\partial_t \equiv t^b \partial_b$)[16]

$$\begin{aligned}
\partial_n\, x^\mu(\sigma) &= -\partial_n\, x^\mu(\sigma + \pi) \,, \\
\partial_t\, x^\mu(\sigma) &= \partial_t\, x^\mu(\sigma + \pi) \,.
\end{aligned} \tag{5.78}$$

[16] Actually, in [81] the *boundary state* representing a closed string disappearing into the vacuum is constructed obeying the boundary conditions

$$\left.\frac{\partial}{\partial \tau} x(\sigma, \tau)\right|_{\tau = T} = 0 \tag{5.76}$$

The *crosscap* boundary state is a modification of it, obtained by the appropriate identification on the boundary, i.e.

$$\begin{aligned}
x(\sigma + \pi, \tau) &= x(\sigma, \tau)|_{\tau = T} \,, \\
\frac{\partial}{\partial \tau} x(\sigma + \pi, \tau) &= -\left.\frac{\partial}{\partial \tau} x(\sigma, \tau)\right|_{\tau = T}
\end{aligned} \tag{5.77}$$

Incidentally, it is quite easy to show that there are no solutions with a plus sign instead of a minus.

When gauging the isometry, there is a covariant generalization of these conditions, namely

$$\begin{aligned} D_n x^\mu(\sigma) &= -D_n x^\mu(\sigma+\pi)\,, \\ D_t x^\mu(\sigma) &= \ D_t x^\mu(\sigma+\pi)\,. \end{aligned} \tag{5.79}$$

Using the value of V_a^{cl} obtained above, we easily find, after fixing the $x^0 = 0$ gauge

$$D_n x^0 = -\frac{1}{k^2}\partial_n x^i k_i + i\,\partial_t \tilde{x}^0\,. \tag{5.80}$$

The antisymmetric tensor and Abelian gauge field backgrounds are projected out from the physical spectrum of unoriented strings in the weak field limit, and so we only consider a nontrivial metric background. This then yields the dual boundary conditions in the form

$$\left(i\,\partial_t\tilde{x}^0 - \frac{1}{k^2}k_j\partial_n x^j\right)(\sigma+\pi) = -\left(i\,\partial_t\tilde{x}^0 - \frac{1}{k^2}k_j\partial_n x^j\right)(\sigma)\,. \tag{5.81}$$

The terms containing the Killing vector cancel away owing to the boundary conditions of the original model; the rest reduces to the orientifold condition on $\tilde{x}^0$,

$$\begin{aligned} \partial_t\tilde{x}^0(\sigma) &= -\partial_t\tilde{x}^0(\sigma+\pi)\,, \\ \partial_n\tilde{x}^0(\sigma) &= \partial_n\tilde{x}^0(\sigma+\pi)\,. \end{aligned} \tag{5.82}$$

The first equation implies $\tilde{x}^0(\sigma) + \tilde{x}^0(\sigma+\pi) =$ constant, so that the crosscap is embedded as a twisted state of the orbifold. This is quite important, because it implies that the orbifold character of the dual target-space is a generic phenomenon, and not a curious peculiarity of toroidal backgrounds. It is curious to remark that the dual manifold always enjoys parity $\tilde{x}^0 \to -\tilde{x}^0$ as an isometry, because $\tilde{g}_{0i} = 0$. The rest of the coordinates still satisfy standard crosscap conditions. An important consequence of this is that at least two points of the boundary are mapped to the orientifold fixed points in the target, which means that local contributions of non-orientable world-sheets are concentrated at the orientifold location; in the bulk of spacetime the dual theory is orientable along the direction $\tilde{x}^0$. This is compatible with the appearance of a non vanishing dual antisymmetric tensor $\tilde{b}_{0i} = g_{0i}/g_{00}$ as long as the original background has a "boost" component. The effects of this background field are supressed only for world-sheets mapped to the fixed point. Another observation is that, in the absence of a $U(1)$ gauge field, there is no collective coordinate for the orientifold, which becomes a rigid object. Indeed, according to the previous formulas, the induced backgrounds are exactly the same as the vacuum dual backgrounds. The theory dual to Type I compactified on a circle of radius R (often called $\tilde{I}$ or I') is then characterized by two orientifold planes and 16 8-branes. Off the D-branes the orientation projection in $\tilde{I}$ does not constrain the local state of the string, meaning that we have a Type II theory, so that the vaccum without branes enjoys N=2 supersymmetry. On the other hand, the state containing the D-brane is only invariant under N=1 supersymmetry, so that it must be a BPS state. This means that it necessarily carries a conserved charge, which in its turn is only possible if this charge is of the RR type, consistent in turn with the $\frac{1}{g}$ behaviour of the D-brane tension. It was the realization of this fact by Polchinski that opened up all recent developments. Please note that from the involutive property of T-duality, $T^2 = 1$, and by interpreting that the 10-dimensional SO(32) Type I strings have 32 D-9-branes, (filling the space) on which open strings can end we get the general rule that *T-duality along a tangent direction maps D-p-branes into D-(p-1)-branes* whereas *T-duality along a normal direction maps D-(p+1)-branes into D-p-branes.* This then means that all branes are, in principle, related through T-duality.

5.4 Physics on the brane (Born-Infeld) versus branes as sources

It is not difficult to study the conditions for conformal invariance of string theory in arbitrary backgrounds with k Dirichlet boundary conditions, which we represent, following Leigh [80], as

$$x^\mu|_{\partial\Sigma} = f^\mu(\eta^A) \qquad\qquad (A = 1, \ldots, 26-k)\,, \tag{5.83}$$

where the boundary of the worldsheet, $\partial\Sigma$, is imbedded into a $(26-k)$ submanifold $\mathcal{M}$, with coordinates η^A, of the target space. This condition is sufficient to ensure that the variation $\delta x^\mu \mid_{\partial\Sigma}$ is tangent to $\mathcal{M}$, thus imposing the k Dirichlet conditions. The action, in the conformal gauge, reads

$$\begin{aligned} S = {} & \frac{1}{4\pi}\int_\Sigma (g_{\mu\nu}\eta^{ab} + i\, b_{\mu\nu}\epsilon^{ab})\,\partial_a x^\mu \partial_b x^\nu + \partial^a\rho\partial_a x^\mu\partial_\mu\phi(x) + \\ & + \frac{i}{2\pi}\int_{\partial\Sigma}(\nu_\mu(x)\partial_n x^\mu - iA_B(\eta)\partial_\tau\eta^B), \end{aligned} \tag{5.84}$$

where ρ is the conformal factor of the 2-dimensional metric, the ν's are k fields perpendicular to $\mathcal{M}$, as defined by $\nu_\mu f^\mu(\eta) = 0$, A is a $U(1)$ field tangent[17] to $\mathcal{M}$ and ϕ, $g_{\mu\nu}$ and $b_{\mu\nu}$ are the usual background dilaton, metric and Kalb-Ramond field. By using Riemann normal coordinates ξ^μ on spacetime, and ζ^A on M, the β-functions can be obtained by a slight modification of the calculation in [81]

$$\begin{aligned} \beta_B(A) = {} & -\frac{1}{2}(B+F)^C_B\partial_C\phi + J^{AC}(B+F)_{AB;C} \\ & + J^{AC}\left(\frac{1}{2}(B+F)^D_B H_{DAE}(B+F)^E_C + K^\mu_{BC}b_{\mu\nu}f^\nu_{,C}\right). \end{aligned} \tag{5.85}$$

[17] Although it is introduced here as a Lagrange multiplier field, it is nowadays seen as the background field of the open string sector living on the D-brane.

Here $J^{AB} = (h - (B+F)^2)^{-1|AB}$, and the normal coordinates expand as $\xi^\mu|_{\partial\Sigma} = f^\mu_A \zeta^A + \frac{1}{2} K^\mu_{AB} \zeta^A \zeta^B + \cdots$ The other β-functions are

$$\beta_\mu(\nu) = \frac{1}{2}\partial_\mu \phi + J^{AC}\left(\frac{1}{2}(B+F)^B_C H_{\mu AB} - K_{\mu AC}\right). \qquad (5.86)$$

It can be shown that the Dirac-Born-Infeld action [95]

$$S_{DBI} = T' \int d^{26-k}\eta e^{-\phi/2}\sqrt{det(h+B+F)}, \qquad (5.87)$$

gives equations of motion which are proportional to the beta functions above. This can be easily generalized to the supersymmetric case. The non-Abelian generalization (corresponding to N coincident D-branes) is not known, but it is believed that in the low energy limit, the effective non-Abelian theory on the brane should be the dimensional reduction of $N = 1$ Super-Yang-Mills with gauge group $SU(N)$ in $d = 10$ dimensions, to the appropriate world-volume of the brane, W_p. To be specific

$$S_p = -T_p \int tr \int_{W_p} d^P \xi e^{-\phi/2}\left(F^2_{\mu\nu} + 2(D_\mu X^I)^2 + [X^I, X^J]^2\right), \qquad (5.88)$$

All fields are $N \times N$ matrices; the eigenvalues of the matrices X^I represent the positions of the N $D-(p-1)$ branes, and the $U(1)$ describes the overall center of mass. When studying the field equations of the effective supergravity theory, there are all kinds of extended solutions, which can be grossly classified into *elementary* (if they do not depend on the string coupling g_s at all), or *solitonic* (if the energy goes as $\frac{1}{g_s^2}$); or, finally, *Dirichlet* if their energy scales like $\frac{1}{g_s}$. There is a Hodge duality between these solutions. For example: for N=1 supergravity in d=10 there is the *elementary string* of Dabholkar et al. [82], with a ten dimensional (Einstein frame) metric of the type

$$\begin{aligned} ds^2 &= e^{\frac{3}{4}(\phi-\phi_0)}\eta_{\mu\nu}dx^\mu dx^\nu - e^{-\frac{1}{4}(\phi-\phi_0)}\delta_{mn}dy^m dy^n, \\ B_{01} &= -e^{\phi-\phi_0}, \\ e^{-\phi} &= e^{-\phi_0}\left(1+\frac{k_2}{y^6}\right), \qquad k_2 = \frac{\kappa^2_{10}T_2}{3\Omega_7}e^{3\phi_0/4}. \end{aligned} \qquad (5.89)$$

where $\mu \in (0,1)$; and $m, n \in (1, \ldots, 9)$, T_2 is the string tension and Ω_n is the volume of the n-dimensional unit sphere. Given the fact that the solution has a timelike singularity [25], this BPS solution can be thought of having a delta function singularity at $y = 0$; so it corresponds to an energy-momentum tensor with support on the world-sheet of the string. Correspondingly, there is also a *solitonic fivebrane*; a solution of the source-free field equations of $d = 10$ supergravity alone. The Ansatz is as above, but with $\mu \in (0, \ldots, 5)$; and $m, n \in (6, \ldots, 9)$ There is now a $dH \neq 0$, which means that there is a nontrivial magnetic charge g_6. Explicitly the solution reads

$$\begin{aligned}
ds^2 &= e^{-(\phi-\phi_0)/4}\eta_{\mu\nu}dx^\mu dx^\nu - e^{3(\phi-\phi_0)/4}\delta_{mn}dy^m dy^n\,, \\
H &= 2k_6 e^{\phi_0/4}\,\epsilon_3\,, \\
e^\phi &= e^{\phi_0}\left(1+\frac{k_6}{y^2}\right), \qquad k_6 = \frac{\kappa_{10}g_6}{\sqrt{2}\Omega_3}e^{-\phi_0/4}\,,
\end{aligned} \tag{5.90}$$

where ϵ_n is the normalized volume form on S^n. There is also a dual version in which the rôles of 2 and 5 are reversed (both in the branes and in the supergravity Lagrangians).

6. The web of dualities and the strong coupling limit: back to the beginning?

There is now a certain amount of evidence for different kinds of dualities (See for example [23]), which can be classified, following Schwarz, as *S-dualities*, *T-dualities*, or *U-dualities*. We shall say that two (not neccessarily different) theories, T_1 and T_2 are T-dual, when T_1 compactified at large Kaluza-Klein volume is physically equivalent to T_2 at small Kaluza-Klein volume. If we call t the modulus associated to global variations of the Kaluza-Klein volume, by $Vol \sim e^t$, this implies a relationship of the general form

$$t(1) \;=\; -t(2)\,. \tag{6.1}$$

We have already seen how this comes about in some simple cases from the sigma model approach to string perturbation theory. S-duality, on the other hand, refers to the equivalence of T_1 at small coupling with T_2 at large coupling. It demands for the dilaton something like

$$\phi(1) \;=\; -\phi(2)\,, \tag{6.2}$$

and, by definition, lies beyond the possibilities of verification by means of perturbation theory. U-duality is a kind of mixture of the two, and claims an equivalence of T_1 at large coupling with T_2 at small Kaluza-Klein volume. In terms of fields,

$$\phi(1) \;=\; \pm t(2)\;. \tag{6.3}$$

In these notes we shall only examine a few representative examples of this web. It is still too early to assess the real meaning of this enormous symmetry.

6.1 S-Duality for the heterotic string in $M_4 \times T_6$

Let us summarize here the clear analysis by A. Sen [83], reporting mostly on joint work with J. Schwarz. We shall present the two existing pieces of evidence, namely, the effective low-energy field theory, and the spectrum of masses and charges of those states which are protected by supersymmetry from receiving quantum corrections. We start from the ten-dimensional action, which is the bosonic part of the effective field theory of the heterotic string

$$S^{het}_{(d=10)} = \frac{1}{32\pi}\int d^{10}x\sqrt{-G^{(10)}}e^{-\phi^{(10)}}\Big[R^{(10)} - G^{(10)\mu\nu}\partial_\mu\phi^{(10)}\partial_\nu\phi^{(10)} + \\ + \frac{1}{12}H^{(10)}_{\mu\nu\rho}H^{(10)\mu\nu\rho} - \frac{1}{4}F^{(10)I}_{\mu\nu}F^{(10)I\mu\nu}\Big], \quad (6.4)$$

where $I = 1, \ldots, 16$ represent the Abelian fields in the Cartan subalgebra of either $E_8 \times E_8$ or SO(32), which are the only ones which *generically* will remain massless upon compactification to four dimensions. Upon the simplest toroidal compactification, the effective four-dimensional theory of the massless modes will be

$$S_{(4)} = \frac{1}{32\pi}\int d^4x\sqrt{-G}e^{-\phi}\Big[R(G) - G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{12}G^{\mu\alpha}G^{\nu\beta}G^{\rho\gamma}H_{\mu\nu\rho}H_{\alpha\beta\gamma} - \\ - \frac{1}{4}G^{\mu\alpha}G^{\nu\beta}F^{(a)}_{\mu\nu}(LML)_{ab}F^{(b)}_{\alpha\beta} - \frac{1}{8}G^{\mu\nu}tr(\partial_\mu ML\partial_\nu ML)\Big]. \quad (6.5)$$

Here the indices $a, b = 1, \ldots, 28$, where the 28 is gotten from the 16 that already existed in $d = 10$, plus another 6 coming from the metric $G^{(10)}$ compactified on T_6, plus another 6 coming from the $B^{(10)}$. The scalar fields have been conveniently packed into a matrix $M \in O(6, 22)$, and the numerical matrix L is given by

$$L = \begin{pmatrix} 0 & 1_6 & 0 \\ 1_6 & 0 & 0 \\ 0 & 0 & -1_{16} \end{pmatrix} \quad (6.6)$$

T-duality in this language is particularly transparent: any $g \in O(6, 22)$ (i.e. such that $g^T Lg = L$) acts by

$$M \to gMg^T\,, \\ A^a_\mu \to g^a{}_b A^b_\mu \quad (6.7)$$

(the rest of the fields being inert under T-duality). The preceding four-dimensional action was written in the String frame. It can be rewritten in the Einstein frame through the rescaling

$$g_{\mu\nu} \equiv e^{-\phi}G_{\mu\nu}\,. \quad (6.8)$$

It is also convenient to introduce the axion field, the Hodge dual of the Kalb-Ramond field

$$H^{\mu\nu\rho} = -\frac{1}{\sqrt{-g}}e^{2\phi}\epsilon^{\mu\nu\rho\sigma}\partial_\sigma\psi\,. \quad (6.9)$$

The dilaton and the axion together constitute a complex scalar field

$$\lambda \equiv \psi + ie^{-\phi}\,. \quad (6.10)$$

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It is then a simple matter to check that the *equations of motion* are invariant under $g \in SL(2,\mathbb{R})$, characterized by four real numbers such that $ac - bd = 1$, and constituting what is called an *S-duality* transformation [84]

$$\lambda \to \frac{a\lambda + b}{c\lambda + d}\,,$$
$$F^{(a)}_{\mu\nu} \to (c\lambda_1 + d)F^{(a)}_{\mu\nu} + c\lambda_2 (ML)_{ab}\tilde{F}^{(b)}_{\mu\nu}\,, \tag{6.11}$$

with all other fields remaining inert under S-duality, and we have used $\lambda \equiv \lambda_1 + i\lambda_2$. The action in the Einstein frame is given by

$$S^{(E)}_{(4)} = \frac{1}{32\pi}\int d^4x\sqrt{-g}\Big[R(g) + \frac{1}{2\lambda_2^2}g^{\mu\nu}\partial_\mu\lambda\partial_\nu\bar{\lambda} - \frac{1}{4}\lambda_2 F^{(a)}_{\mu\nu}(LML)_{ab}F^{(b)\mu\nu}$$
$$+\frac{1}{4}\lambda_1 F^{(a)}_{\mu\nu}L_{ab}\tilde{F}^{(b)\mu\nu} - \frac{1}{8}g^{\mu\nu}tr(\partial_\mu ML\partial_\nu ML)\Big]\,. \tag{6.12}$$

It is a characteristic feat that this action is *not* S-dual invariant; only the equations of motion enjoy this property. It is also characteristic that there exists a dual form of the ten-dimensional effective action (known already from supergravity), using the seven form $(*H)_7$ instead of the three-form H_3. The effective four-dimensional action it implies is manifestly S-dual, although only the equations of motion are then T-duality invariant. Another important feat is that quantum effects associated to the term $\lambda_1 F^{(a)}_{\mu\nu}L_{ab}\tilde{F}^{(b)\mu\nu}$ break the classical SL$(2,\mathbb{R})$ down to SL$(2,\mathbb{Z})$ (because λ_1 acts as a generalized θ angle). Let us now consider the spectrum of charged particles in the theory. In the presence of a current $J^{(a)}_\mu$, whose conserved charge is defined by $e^{(a)} \equiv \int d^3x\sqrt{g}j^{(a)0}$, the asymptotics of the radial electric fields changes to $F^{(a)}_{0r} \sim \frac{q^{(a)}_{el.}}{r^2}$ Using the equations of motion it can be shown that $q^{(a)}_{el.} = \frac{1}{\lambda_2^{as}M^{as}_{ab}e^{(b)}}$, where *as* stands for the asymptotic value. We know, on the other hand [72], that $e^{(a)} = \alpha^{(a)} \in \Lambda^{Narain}$, where Λ^{Narain} is an even, sel-dual, Lorentzian lattice with metric L. Elementary strings states do not have any magnetic charge, but other states will. The Dirac quantization condition [31] then forces $q^{(a)}_{mag.} = L_{ab}\beta^{(b)}$ where $\beta^{(b)} \in \Lambda^{Narain}$. Taking into account the modification of the quantization conditions in the presence of a θ angle (Witten effect)[86], the final allowed spectrum is

$$(q^{(a)}_{el}, q^{(a)}_{mag}) \equiv \left(\frac{1}{\lambda_2^{as}}M^{as}_{ab}(\alpha^{(b)} + \lambda_1^{as}\beta^{(b)})\right), L_{ab}\beta^{(b)})\,, \tag{6.13}$$

which is easily seen to be invariant under both SL$(2,\mathbb{Z})$ and O$(6,22;\mathbb{Z})$. A similar analysis shows that the masses of those particles sitting in short multiplets of the supersymmetry algebra obey the formula

$$m^2 \equiv \frac{1}{16}(\alpha^{(a)}\beta^{(a)})\mathcal{M}^{as}(M^{as} + L)_{ab}\begin{pmatrix}\alpha^{(b)}\\ \beta^{(b)}\end{pmatrix}\,, \tag{6.14}$$

where the matrix $\mathcal{M}$ is given by

$$\mathcal{M} \equiv \frac{1}{\lambda_2} \begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & |\lambda|^2 \end{pmatrix}, \tag{6.15}$$

which is, again, invariant under both $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{O}(6,22;\mathbb{Z})$.

6.2 The strong coupling limit of IIA strings, SL$(2,\mathbb{Z})$ duality of IIB strings and heterotic/Type I duality

1. If we are willing to make the hypothesis that supersymmetry is not going to be broken whilst increasing the coupling constant, g_s, some astonishing conlusions can be drawn. Assuming this, massless quanta can become massive as g_s grows only if their number, charges and spins are such that they can combine into massive multiplets [87] (which are all larger than the irreducible massless ones). The only remaining issue, then, is whether any other massless quanta can appear at strong coupling. Now, as we have seen, in the IIA theory there are states associated to the RR one form, A_1, namely the D-0-branes, whose tension goes as $m \sim \frac{1}{g_s}$. This clearly gives new massless states in the strong coupling limit. There are reasons[18] to think that this new massless states are the first level of a Kaluza-Klein tower associated to compactification on a circle of an 11-dimensional theory. Actually, assuming an 11-dimensional spacetime with an isometry $k = \frac{\partial}{\partial y}$, an Ansatz which exactly reproduces the dilaton factors of the IIA string is

$$ds^2_{(11)} = e^{\frac{4}{3}\phi}(dy - A^{(1)}_\mu dx^\mu)^2 + e^{-\frac{2}{3}\phi} g_{\mu\nu} dx^\mu dx^\nu \,. \tag{6.16}$$

Equating the two expressions for the D0 mass,

$$\frac{1}{g_s} = \frac{1}{R_{11}}\,, \tag{6.17}$$

leads to $R_{11} = e^{\frac{2}{3}\phi} = g_A^{2/3}$.

2. All supermultiplets of massive one-particle states of the IIB supersymmetry algebra contain states of at least spin 4. This means that under the previous set of hypothesis, the set of massless states at weak coupling must be exactly the same as the corresponding set at strong coupling. This means that there must be a symmetry mapping weak coupling into strong coupling. There is a well-known candidate for this symmetry: let us call, as usual, l the RR scalar and ϕ the dilaton (NSNS). We can pack them together into complex scalar

$$S = l + i e^{-\frac{\phi}{2}}\,. \tag{6.18}$$

[18] In particular: the fact that there is the possibility of a central extension in the IIA algebra, related to the Kaluza-Klein compactification of the d=11 Supergravity algebra.

The IIB supergravity action in d=10 is invariant under the SL$(2,\mathbb{R})$ transformations

$$S \to \frac{aS+b}{cS+d}, \tag{6.19}$$

if at the same time the two two-forms, $B_{\mu\nu}$ (the usual, ever-present, NS field), and $A^{(2)}$, the RR field transform as

$$\begin{pmatrix} B \\ A^{(2)} \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B \\ A^{(2)} \end{pmatrix}, \tag{6.20}$$

Both the, Einstein frame, metric $g_{\mu\nu}$ and the four-form $A^{(4)}$ are inert under this SL$(2,\mathbb{R})$ transformation. A discrete subgroup SL$(2,\mathbb{Z})$ of the full classical SL$(2,\mathbb{R})$ is believed to be an exact symmetry of the full string theory. The exact imbedding of the discrete subgroup in the full SL$(2,\mathbb{R})$ depends on the vacuum expectation value of the RR scalar. The particular transformation

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.21}$$

maps ϕ into $-\phi$ (when $l=0$), and B into $A^{(2)}$. This is then an S-duality type of transformation, mapping the ordinary string with NS charge, to another string with RR charge (which then must be a D-1-brane, and is correspondingly called a *D-string*), and, from there, is connected to all other D-branes by T-duality. Using the fact that upon compactification on S^1, IIA at R_A is equivalent to IIB at $R_B \equiv 1/R_A$, and the fact that the effective action carries a factor of $e^{-2\phi}$ we get

$$R_A g_B^2 = R_B g_A^2, \tag{6.22}$$

which combined with our previous result, $g_A = R_{11}^{3/2}$ implies that $g_B = \frac{R_{11}^{3/2}}{R_A}$. Now the Kaluza-Klein Ansatz implies that from the eleven dimensional viewpoint the compactification radius is measured as

$$R_{10}^2 \equiv R_A^2 e^{-2\phi/3}, \tag{6.23}$$

yielding

$$g_B = \frac{R_{11}}{R_{10}}. \tag{6.24}$$

3. From the effective actions written above it is easy to check that there is a (S-duality type) field transformation mapping the SO(32) Type I open string into the SO(32) Heterotic one namely

$$\begin{aligned} g_{\mu\nu} &\to e^{-\phi} g_{\mu\nu}^{Het}, \\ \phi &\to -\phi, \\ B' &\to B. \end{aligned} \tag{6.25}$$

This means that physically there is a strong/weak coupling duality, because coupling constants of the compactified theories would be related by

$$\begin{aligned} g_{het} &= 1/g_I \,, \\ R_{het} &= R_I/g_I^{1/2} \,. \end{aligned} \tag{6.26}$$

6.3 Statistical interpretation of the black hole entropy

The fact that the area of (the horizon of) a black hole can be interpreted as a kind of entropy was actually first discovered through an analogy between the equations of black hole physics and the equations of ordinary thermodynamics [99]. Hawking's astonishing discovery that even the '*dead*' Schwarzschild black hole radiates with a black body spectrum led to a much firmer identification of the entropy as

$$S_{BH} \equiv \frac{1}{4}\frac{A}{G_4}\hbar \tag{6.27}$$

where A is the area of the black hole's horizon and G_4 is the four-dimensional Newton constant. Furthermore one finds in case of the four-dimensional Schwarzschild black hole that

$$A \equiv 4\pi(G_4 M)^2 \,, \tag{6.28}$$

where M is the mass of the black hole. The problem as to whether a statistical interpretation of this entropy (as the logarithm of a corresponding density of states) exists at all is undoubtebly one of the most important open problems in the whole topic of gravitational physics. Recently (cf. [100] for an introductory review), there has been some progress in understanding the counting of states, albeit not in the physically most interesting cases, but rather for *extremal* black holes; that is, holes such that the charge is as big as it can be in a way consistent with the Cosmic Censorship Hypothesis (that is, without creating a naked singularity). These black holes (which can often be considered BPS states) are usually uninteresting, because they have zero Hawking temperature and, in addition, those which can be embedded into the low energy limit of string theory, usually have singular dilaton behaviour. But it was pointed out in [101] that in some cases, with several charges, the horizon stays nonsingular. It is exactly for this case that one can give a *microscopical* interpretation of the black hole entropy. The main idea which makes the counting feasible is first of all, the fact that, as stems easily from eq. (2.100), the Newton constant in d dimensions is given by

$$G_d \sim \frac{g_s^2 l_s^8}{V_{10-d}} \,, \tag{6.29}$$

where numerical factors have been ignored. This fact on itself means that the strength of the gravitational coupling, measured roughly by

$$G_d M \sim g_s^2 M \tag{6.30}$$

is small when $g_s \to 0$ as long as the mass does not grow faster than g_s^{-2}. This in its turn implies that there must exist some weak coupling description of these states which clearly will consist in an appropiate set of D-branes with corresponding RR charges.[19] Their BPS property implies that, as we make the string coupling constant g_s grow, the degeneracy of the states does not change. But in doing so, we change from a perturbative description in terms of branes, to a non-perturbative black hole configuration.[20] This is the first time a statistical interpretation of the entropy of a black hole in more than three dimensions is obtained, and as such, is one of the most important applications of D-brane physics.[21] There are, however, essential complications to treat non extremal black holes using this set of ideas (except in the case where they are close to extremality). A useful, quite detailed, general reference is [103].

7. Concluding remarks

It is probably fair to say that most fundamental questions on quantum gravity are still waiting to be answered. Many previously unsuspected relationships between ordinary gauge theories and gravity are now appearing, however, and, everything points in the direction of a much more unified and symmetric fundamental theory than was previously thought to be the case. It can only be hoped that specific and concrete experimental predictions of the theory can be made in the near future.

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[19]Technical complications related to the necessity (briefly alluded to above) of having a finite-area horizon in the extremal limit imply that in four dimensions one needs at least four different charges.

[20]The turning point being clearly when the curvature is of the order of l_s^{-2}.

[21]It is ironic to remark in this respect that shortly *before* the statistical interpretation was proposed, it was *proven* (using euclidean regularity arguments) [102] that extremal black holes should have zero entropy even in those cases in which they enjoyed non-zero area.

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